

The Pontrjagin-Hopf invariants for Sobolev maps

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Abstract

1 Introduction

Variational problems with topological constraints can exhibit many interesting mathematical properties. There are many open mathematical questions in this area. The diverse properties these problems possess give them the potential for different applications. We find variational problems with topological constraints in high energy physics, hydrodynamics, and material science, [9, 3, 48]. Usually, the functional to be minimized represents energy and the topological constraint is that the map must be in a fixed homotopy class. An energy functional dictates a natural class of maps – the finite energy maps. For some models finite energy maps include discontinuous maps and this introduces new technical difficulties into the homotopy theory. Many traditional arguments have been designed for continuous or smoother maps and do not work for finite energy maps. Thus, one has to look for new approaches and new interpretations that survive the lack of regularity. This may lead to new geometric results and require more subtle analytic techniques. Even if all finite energy maps were continuous, one would want analytic expressions for complete homotopy invariants in order to apply the direct method in the calculus of variations. This is another challenge in geometry.

This is a relatively new area of research with interesting interplay between geometry and analysis. The first steps have been made by studying the homotopy classes for Sobolev maps, see [17, 49, 10, 11, 15, 14, 27]. One can view Sobolev maps as the maps with finite Sobolev energy aka the Sobolev norm. These studies are very important because Sobolev spaces are a basic technical tool in analysis.

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We were originally drawn into this area of research from our study of the Skyrme and Faddeev models. These are non-linear sigma models from high-energy physics [42, 43, 44, 19, 20]. The Faddeev-Skyrme model has been generalized to encompass maps between general Riemannian manifolds. The original Skyrme model was for maps from \mathbb{R}^3 to the 3-sphere, and the original Faddeev model was for maps from \mathbb{R}^3 to the 2-sphere.

In this paper we concentrate on maps from \mathbb{R}^3 or a closed orientable 3-manifold to the 2-sphere. (Maps into the 3-sphere are easier to analyze.) Continuous maps of this type are classified up to homotopy by a primary invariant and a secondary invariant. In [6] we gave the first analytical description of the secondary invariant for smooth maps. In this paper we extend the primary and secondary invariants to appropriate Sobolev maps and finite Faddeev energy maps and establish the properties of these invariants expected from the smooth case. For continuous functions, the primary invariant can be viewed as an obstruction to a certain lifting problem. We turn this around and use the lifting problem to define the primary invariant for sufficiently regular (but possibly discontinuous maps). In traditional homotopy theory, the homotopy class of a map is often studied by attempting to lift the map to higher levels of a Whitehead or a Postnikov tower. The lifting results in this paper can be viewed as a first step generalizing these techniques to Sobolev maps.

Incidentally, lifting maps to maps taking values in a larger space has the effect of replacing a variational problem with a new variational problem with an additional differential equation constraint. The hope is that it will be easier to analyze the new problem and constraint than it is to analyze the old problem. This is indeed the case with the Faddeev functional as we showed in [6].

We review the homotopy classification of S^2 -valued maps in Section 2.1 and the definition of the Faddeev and Skyrme functionals in Section 2.3 below. The local lifting result appears in Section 3. It may be viewed as a non-linear analog of the Poincaré lemma. This local result leads to the global lifting criteria in Section 4. The local lifting result also leads to a definition of the primary invariant valid for finite energy maps. The secondary invariant compares two maps with the same primary invariant. Two such maps are related by an intertwining map. For the continuous case the degree of this intertwining map determines whether the two maps are homotopic. We extend this to finite energy maps by proving that the integral expression for the degree of such an intertwining map is an integer. We also show that changing the intertwining map changes this degree by an even multiple of the divisibility of the primary invariant as expected from the continuous case. Our results will generalize to other spaces of maps with similar or better regularity.

From what we prove it follows, in particular, that for any finite energy map there is a smooth map with the same invariants, and that two smooth maps have the same invariants if and only if the maps are homotopic.

We reconsider minimization of the Faddeev functional at the end of this paper. We have made an effort to make this paper accessible to analysts, topologists and mathematical physicists. The general outline of the paper can be gleaned from the following table of contents:

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2 Review and examples

This section contains a brief review of background material and several examples.

2.1 Homotopy classification of S^2 -valued maps

The homotopy classification of maps $M \rightarrow S^2$ has been known since the nineteen-thirties. First Hopf [30] classified maps from S^3 into S^2 and defined a complete integer-valued invariant that counts the linking number of the inverse image of a pair of regular values. In

modern notation the result of Hopf is $\pi_3(S^2) = \mathbb{Z}$. Shortly thereafter Pontrjagin [39] studied the general case of maps from a three-dimensional simplicial complex into S^2 . When the three-dimensional complex is an orientable manifold the result, Pontrjagin writes, ‘may be formulated by means of the usual homologies, which presents a certain advantage’ ([39, §4]). From this point forward we will restrict our attention to closed orientable 3-manifolds.

To set some useful notation we start by reviewing the easier case of the homotopy classification of S^3 -valued maps. A map $u : M \rightarrow S^3$ induces a map on the top cohomology $u^* : H^3(S^3; \mathbb{Z}) \rightarrow H^3(M; \mathbb{Z})$. Since the top cohomology of an oriented manifold is canonically isomorphic to \mathbb{Z} the map u^* is just multiplication by an integer. This integer is called the degree of the map and it is denoted by $\deg u$. The map u is classified up to homotopy by its degree. The intersection theory interpretation of $\deg u$ is the number of inverse images of a regular point counted with sign. Using the deRham model one can describe the degree of the map u as the unique integer such that

$$\int_M u^* \alpha = (\deg u) \int_{S^3} \alpha,$$

for all top dimensional forms α on S^3 . In particular one has

$$\deg u = \int_M u^* \omega_{S^3},$$

where ω is a normalized volume form on S^3 , i.e., $\int_{S^3} \omega_{S^3} = 1$. It is well-known but not obvious that the three descriptions of the degree given above coincide [13]. The last description makes sense for possibly discontinuous functions provided that the integral converges, however it is far from obvious that the integral would still be an integer (in fact, sometimes it is not). We will come back to this point in great detail later.

Returning to the case of maps from S^3 to S^2 we see that the Hopf invariant also has multiple descriptions. The description we gave as the linking number of the preimages of two regular values is the one arising from intersection theory. If a map $\varphi : S^3 \rightarrow S^2$ is sufficiently regular, there is a way to actually compute the Hopf invariant of φ by evaluating a certain integral. This was found by J. H. C. Whitehead [50] and goes as follows. (See [13] as well.) The pull-back of the normalized volume form, $\varphi^* \omega_{S^2}$, is a closed 2-form on S^3 , and, since $H^1(S^3) = 0$, this form is exact, i.e., $\varphi^* \omega_{S^2} = d\theta$, for some 1-form θ . The result of Whitehead is

$$\text{Hopf}(\varphi) = \int_{S^3} \theta \wedge d\theta. \quad (1)$$

This is the expression analogous to the integral for the degree of the map. In fact it is a de Rham description of a difference cocycle arising from obstruction theory. In fact as we will explain in the next subsection the Hopf invariant can be described as the degree of a related map.

The final remark we should make about the Hopf invariant is that since one can pick any normalized volume form ω_{S^2} and any form θ satisfying $\varphi^* \omega_{S^2} = d\theta$, one can pick nice forms. If one fixes a Riemannian metric on S^3 , then among all θ satisfying $\varphi^* \omega_{S^2} = d\theta$ there is

a unique one, θ^φ , such that $\delta\theta^\varphi = 0$, where δ is the adjoint of d on 1-forms. This will be important when we generalize this invariant to possibly discontinuous finite-energy maps.

2.1.1 The classification

Now return to the homotopy classification of maps from an arbitrary 3-manifold M to S^2 . Compared to the $S^3 \rightarrow S^2$ case, two new features arise. First, there is a new invariant given by the induced map on second cohomology. Second, the Hopf invariant generalizes into a secondary invariant. The secondary invariant is an invariant of a pair of maps with the same primary invariant. It sometimes takes values in a finite cyclic group. The Hopf invariant of a map is the secondary invariant of the pair consisting of the map and a constant map.

Theorem 1 (Pontrjagin) *Let M be a closed, connected, oriented three-manifold. To any continuous map φ from M to S^2 one associates the pull-back $\varphi^*\mu_{S^2} \in H^2(M; \mathbb{Z})$ of the orientation class $\mu_{S^2} \in H^2(S^2; \mathbb{Z})$. Every cohomology class in $H^2(M; \mathbb{Z})$ may be obtained from some map, and two maps with different classes lie in different homotopy classes. The homotopy classes of maps with a fixed class $\alpha \in H^2(M; \mathbb{Z})$ are in bijective correspondence with $H^3(M; \mathbb{Z})/(2 \cdot \alpha \cup H^1(M; \mathbb{Z}))$.*

For integral homology 3-spheres (i.e. closed three-manifolds M with $H_1(M) = 0$), the homotopy classes of maps $M \rightarrow S^2$ are still completely characterized by the Hopf number, which in the case of sufficiently regular φ can be computed via the same formula (1). If $H_1(M) \neq 0$, then the situation can be more complicated. From Pontrjagin's description we see that the primary invariant, α , is defined for individual maps as $\alpha = \varphi^*\mu_{S^2}$. If $\alpha = 0$, then

$$H^3(M; \mathbb{Z})/(2 \cdot \alpha \cup H^1(M; \mathbb{Z})) = H^3(M; \mathbb{Z}) = \mathbb{Z},$$

and this is still the Hopf invariant. If α is not trivial and is not pure torsion, then one must construct a secondary invariant. The secondary invariant is a relative invariant, meaning that it tells whether two maps with the same primary invariant are homotopic or not. The homotopy invariants were originally discovered in the framework of obstruction theory or intersection theory.

Using Poincaré duality we may identify the second cohomology with the first homology to see the intersection theory interpretation. Under this identification the primary invariant is just the homology class of the inverse image of a regular point. The secondary invariant is the relative framing of the inverse image under the first map with respect to the inverse image under the second map. More precisely, since we are assuming that the two maps have the same primary invariant there is an oriented surface connecting the inverse images of the two regular points. The relative framing counts the number of intersections of this surface with the inverse images of two more regular points, one for each map. See [8] for further exposition on this point. For integral homology spheres the secondary invariant is just the Hopf invariant which geometrically is the linking number of a pair of regular values.

In the framework of obstruction theory the primary invariant is the obstruction to lifting the map $\varphi : M \rightarrow S^2$ to a map $\Phi : M \rightarrow S^3$ such that $\varphi = \sigma \circ \Phi$, where σ is the Hopf map

described later. The secondary invariant is just the class of the difference cocycle. When the primary invariant vanishes so that there is a lift, the secondary invariant reduces to the degree of the lift Φ .

It is instructive to consider a few examples. One could skip to the examples in Section 2.1.5 now. Before presenting the examples we review the definition of the quaternions and several models of S^2 that make it easier to construct examples. We also give a more analytical description of the homotopy classification.

2.1.2 Geometry of the quaternions

Many important maps and forms related to the homotopy classification of maps between a 3-manifold and S^2 can be expressed in a compact form using quaternionic notation. We denote a generic quaternion by $q = q^0 + q^1 \mathbf{i} + q^2 \mathbf{j} + q^3 \mathbf{k}$ and call q^0 the real part of q and $q^1 \mathbf{i} + q^2 \mathbf{j} + q^3 \mathbf{k}$ the imaginary part of q . Multiplication is specified by requiring the quaternions to be a unital associative algebra with

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Changing the sign of the imaginary part gives the conjugate \bar{q} of q , and one can check that $\overline{\bar{p}q} = \bar{q}\bar{p}$ on a basis. For *purely imaginary* quaternions, i.e., those with the real part zero, we have $\bar{p} = -p$.

- The set of all quaternions is denoted by \mathbb{H} .

The usual inner product on \mathbb{H} is

$$\langle p, q \rangle = \frac{1}{2} (\bar{p}q + \bar{q}p) = p^0 q^0 + p^1 q^1 + p^2 q^2 + p^3 q^3.$$

We denote the corresponding norm by $|p| = \langle p, p \rangle^{1/2}$.

- The unit 3-sphere S^3 is identified with the unit (norm 1) quaternions.

This is a Lie group sometimes denoted $\mathrm{Sp}(1)$. Its Lie algebra, $\mathfrak{sp}(1)$, can be identified with the space of purely imaginary quaternions, \mathbb{R}^3 , with the Lie bracket $[p, q] = pq - qp$. We will often use x to denote a purely imaginary quaternion. Viewing x and y as vectors in \mathbb{R}^3 , we see that $\frac{1}{2}[x, y]$ is the usual cross-product $x \times y$. Also, if, in addition, x has norm one, then there is a useful decomposition of y into a component parallel to x and a component perpendicular to x :

$$y = \langle y, x \rangle x + \frac{1}{2} x [y, x]. \quad (2)$$

- The complex numbers embed into the quaternions as $a + b\mathbf{i}$. The set of quaternions \mathbb{H} becomes a complex vector space with \mathbb{C} acting on the left. This identifies \mathbb{C}^2 with \mathbb{H} via $(z, w) \mapsto z + w\mathbf{j}$. Since every complex vector space has a canonical orientation this induces an orientation on \mathbb{H} . The resulting complex orientation is given by $dq^0 \wedge dq^1 \wedge dq^2 \wedge dq^3$.

The group of unit quaternions can be identified with the group of special unitary matrices $\mathrm{SU}(2)$ via the identification of \mathbb{H} with \mathbb{C}^2 . This association takes any unit quaternion q to

the unitary map $p \mapsto p\bar{q}$. Here we take the conjugate of q so that the $SU(2)$ action is a left action. More concretely we have

$$\mathbf{i} \mapsto \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \mathbf{j} \mapsto \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{k} \mapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Other identifications are possible as well. The Lie algebra $\mathfrak{sp}(1)$ can be identified with the Lie algebra $\mathfrak{su}(2)$ via the complex vector space structure on \mathbb{H} used above.

Some people prefer to think about $SU(2)$, others about S^3 and others about $Sp(1)$. Of course all three are just different names for the same thing. We choose to use quaternionic notation because it is shorter.

- We identify the usual sphere S^2 with the unit sphere in the space of purely imaginary quaternions.
- We identify S^1 with the unit quaternions of the form $q^0 + q^1 \mathbf{i}$. Thus $S^2 \subset S^3$, $S^1 \subset S^3$, and $S^2 \cap S^1 = \mathbf{i} \cup -\mathbf{i}$.

A quaternion-valued differential form is a combination

$$a = a^0 + a^1 \mathbf{i} + a^2 \mathbf{j} + a^3 \mathbf{k},$$

and quaternionic conjugation and multiplication (therefore the inner product and commutator) extend to quaternion-valued differential forms. The extension of conjugation is obvious and the multiplication of such forms is given by:

$$(a \wedge b)(X_1, \dots, X_k, X_{k+1}, \dots, X_{k+\ell}) := \frac{1}{k!\ell!} \sum_{\sigma} (-1)^{\sigma} a(X_{\sigma(1)}, \dots, X_{\sigma(k)}) b(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)}).$$

If α and β are real differential forms and p and q are quaternions then the wedge-product of the quaternion-valued forms $p\alpha$ and $q\beta$ is given by $(p\alpha) \wedge (q\beta) = pq\alpha \wedge \beta$ and extending this linearly one recovers the definition of the quaternionic wedge-product of forms.

Remark 2 Throughout the paper we will use ω_M to denote a normalized volume form on a manifold M . Given a Riemannian metric on an oriented manifold M there is a unique compatible volume form denoted by $d\text{vol}_M$ and specified by $d\text{vol}_M(e_1, \dots, e_n) = 1$ for any oriented orthonormal basis $\{e_k\}$. A normalized volume form can then be constructed by dividing by the volume of the manifold. Applying this procedure to the low-dimensional unit spheres using the induced metrics and orientations (outer normal first convention) gives the forms defined below.

Definition 3 The normalized volume forms on S^1 , S^2 and S^3 are

$$\begin{aligned} \omega_{S^1} &= \frac{1}{2\pi} (x^1 dx^2 - x^2 dx^1) \\ \omega_{S^2} &= \frac{1}{4\pi} (x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2) \\ \omega_{S^3} &= \frac{1}{2\pi^2} (x^1 dx^2 \wedge dx^3 \wedge dx^4 - x^2 dx^1 \wedge dx^3 \wedge dx^4 + x^3 dx^1 \wedge dx^2 \wedge dx^4 - x^4 dx^1 \wedge dx^2 \wedge dx^3) \end{aligned}$$

In this last definition S^n is the unit sphere in \mathbb{R}^{n+1} with coordinates (x^1, \dots, x^{n+1}) . Using the quaternionic description of the low-dimensional spheres we have the following lemma obtained from direct calculation.

Lemma 4 *The normalized volume form on S^1 is given by*

$$\omega_{S^1} = \frac{1}{2\pi\mathbf{i}} z^{-1} dz. \quad (3)$$

The normalized volume form on S^2 is given by

$$\omega_{S^2} = -\frac{1}{8\pi} x dx \wedge dx. \quad (4)$$

The normalized volume form on S^3 is given by

$$\omega_{S^3} = -\frac{1}{12\pi^2} \operatorname{Re} (q^{-1} dq \wedge q^{-1} dq \wedge q^{-1} dq). \quad (5)$$

Proof. The volume form on S^1 is well known. For S^2 begin by noting that $x dx \wedge dx$ is real. To see this note that $|x| = 1$ and $x + \bar{x} = 0$ implies that $x^2 = -1$, so $x dx + dx x = 0$ and

$$\overline{x dx \wedge dx} = \overline{dx \wedge dx} \bar{x} = -\overline{dx} \wedge \overline{dx} (-x) = dx \wedge dx x = -dx \wedge x dx = x dx \wedge dx.$$

Since $x dx \wedge dx$ is real we can expand it keeping track of just the real products to obtain the result.

For the S^3 -case, notice that $|q| = 1$ implies that $q^{-1} dq$ is purely imaginary and $q^{-1} dq$ is invariant under left multiplication by constants from S^3 . It follows that we can just do the comparison at $q = 1$ where $q^{-1} dq = dq^1 \mathbf{i} + dq^2 \mathbf{j} + dq^3 \mathbf{k}$. \square

It is well known that the quotient $S^1 \backslash \operatorname{Sp}(1)$ is homeomorphic to S^2 . In fact the homeomorphism is given by $\widehat{\sigma}([q]) = q^{-1} \mathbf{i} q$. The double cover of the three-dimensional rotation group $\operatorname{SO}(3)$ may be identified with $\operatorname{Sp}(1)$. Given a unit quaternion q define a rotation $A_q : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by $A_q(x) = qxq^{-1}$. Every rotation arises from exactly two unit quaternions in this way. This elegant description of the 3-dimensional rotation group is one of the main advantages of quaternionic notation. We see that the homeomorphism $\widehat{\sigma}$ just sends the equivalence class $[q]$ to the image of \mathbf{i} under the rotation associated to q^{-1} .

Notice that the action of S^1 in the above quotient is by constant multiplication. This is different from the S^1 subgroup of S^3 that corresponds to the diagonal matrices in $\operatorname{SU}(2)$. The 2-sphere is also homeomorphic to one-dimensional complex projective space. Recall that one-dimensional complex projective space is $\mathbb{CP}^1 = \mathbb{C}^2 - \{0\} / \sim$ where $(z, w) \sim (z\lambda, w\lambda)$. Equivalence classes are denoted by $[z : w]$. Using the identification of \mathbb{H} with \mathbb{C}^2 , the homeomorphism between $S^1 \backslash \operatorname{Sp}(1)$ and \mathbb{CP}^1 is just the inclusion. Of course \mathbb{CP}^1 is also homeomorphic to the extended complex numbers $\overline{\mathbb{C}}$. Here the identification takes z to $[z : 1]$ and ∞ to $[1 : 0]$. The map from S^3 to S^2 given by composition of projection with the homeomorphism is an important map known as the Hopf map.

Definition 5 *The Hopf map or Hopf fibration $\sigma : S^3 \rightarrow S^2$ is given by $\sigma(q) = q^{-1}\mathbf{i}q$.*

Remark 6 *The Hopf map can be described in several different ways. A second way to describe the same map is as the map from S^3 viewed as the unit sphere in \mathbb{C}^2 to \mathbb{CP}^1 taking (z, w) to $[z : w]$ or, using the extended complex plane, to z/w .*

The next subsection contains a series of computations leading to the Hopf invariant of the Hopf map. Many of the intermediate formulas will be used repeatedly throughout the paper.

2.1.3 Hopf invariant of the Hopf map

The definition of the Hopf invariant does not depend on the choice of normalized volume form. (One can see this by an elementary application of de Rham theory.) We choose the standard volume form.

Lemma 7 *Consider the Hopf map $\sigma : S^3 \rightarrow S^2$ given by $q \mapsto x = q^{-1}\mathbf{i}q$. Define*

$$a = q^{-1}dq$$

and

$$\theta = \frac{1}{2\pi} \langle a, x \rangle = \frac{1}{2\pi} \langle q^{-1}dq, q^{-1}\mathbf{i}q \rangle. \quad (6)$$

Then

$$\sigma^*\omega_{S^2} = d\theta \quad (7)$$

and

$$\omega_{S^3} = -\frac{1}{12\pi^2} \operatorname{Re} (a^{\wedge 3}) = \theta \wedge \sigma^*\omega_{S^2} = \theta \wedge d\theta. \quad (8)$$

Proof. When $x = \sigma(q) = q^{-1}\mathbf{i}q$, we have

$$dx = -q^{-1}dq x + x q^{-1}dq = [x, a]. \quad (9)$$

Then

$$\begin{aligned} -8\pi \omega_{S^2} &= x dx \wedge dx = x[x, a]^{\wedge 2} \\ &= -axa + a^{\wedge 2}x + xa^{\wedge 2} + xaxax = 4 \operatorname{Re} (a^{\wedge 2}x), \end{aligned} \quad (10)$$

where in the last line we used the fact that $x dx \wedge dx$ is real to cycle and combine terms. On the other hand,

$$\begin{aligned} 2\pi d\langle a, x \rangle &= d\langle q^{-1}dq, q^{-1}\mathbf{i}q \rangle = d\langle dq q^{-1}, \mathbf{i} \rangle = \langle dq q^{-1} \wedge dq q^{-1}, \mathbf{i} \rangle \\ &= \langle a^{\wedge 2}, x \rangle = -\operatorname{Re} (a^{\wedge 2}x) = -\frac{1}{4} x dx \wedge dx. \end{aligned} \quad (11)$$

This proves that $d\theta = \sigma^* \omega_{S^2}$. To prove (8) we work backwards with the last equation from Lemma 4, i.e.,

$$\omega_{S^3} = -\frac{1}{12\pi^2} \operatorname{Re} (a^{\wedge 3}) .$$

Splitting a into components parallel and perpendicular to x using equation (2), we compute:

$$a^{\wedge 3} = \left(\langle a, x \rangle x + \frac{1}{2} x[a, x] \right)^{\wedge 3} = \frac{3}{4} \langle a, x \rangle x \wedge [a, x]^{\wedge 2} + \frac{1}{8} (x[a, x])^{\wedge 3} .$$

Here we used the fact that the form $\langle a, x \rangle$ is real and that $[a, x]x = -x[a, x]$ and $x^2 = -1$. A useful observation is that

$$\operatorname{Re} (x[a, x])^{\wedge 3} = 0 ,$$

because we can cycle the terms in the real part without picking up signs, and because $x^2 = -1$. It follows that

$$\omega_{S^3} = -\frac{3}{4 \cdot 12\pi^2} \operatorname{Re} (\langle a, x \rangle x \wedge [a, x]^{\wedge 2}) = -\frac{1}{8\pi} x[a, x]^{\wedge 2} \wedge \frac{1}{2\pi} \langle a, x \rangle = \theta \wedge \sigma^* \omega_{S^2} . \quad (12)$$

End of proof. □

Corollary 8 $\operatorname{Hopf}(\sigma) = 1$.

There are a number of formulas that we will use in computations throughout the rest of of this paper. All of them can be verified by direct computation. We have collected them in the following lemma.

Lemma 9 *The following formulas are true when $|q| = 1$ and $x, y \in \mathfrak{sp}(1) \cong \mathbb{R}^3$.*

$$\langle q^{-1}xq, q^{-1}yq \rangle = \langle x, y \rangle, \quad (13)$$

The following formulas are true when $x = q^{-1}\mathbf{i}q$ and $a = q^{-1}dq$.

$$d\langle q^{-1}dq, q^{-1}\mathbf{i}q \rangle = -\frac{1}{4} x dx \wedge dx, \quad (14)$$

$$\langle a, a^{\wedge 2} \rangle = -\operatorname{Re} (a^{\wedge 3}) = -\frac{3}{4} \operatorname{Re} (\langle a, x \rangle x[a, x]^{\wedge 2}). \quad (15)$$

2.1.4 Analytic description of homotopy for maps $M^3 \rightarrow S^2$

Using obstruction theory or intersection theory is not a suitable method to keep track of the homotopy class of a map for the functions with relatively little regularity that are encountered in the variational problems that we considered. As far as we know, until our recent paper [6], there were no analytically suitable tools to distinguish all homotopy classes of maps from M to S^2 even for simple examples such as $S^2 \times S^1$, T^3 and S^3/\mathbb{Z}_2 . Developing suitable tools was one of the goals of our previous paper [6]. That paper also contains a new proof of Pontrjagin's theorem. The tools developed in [6] were sufficient to establish the existence of minimizers of the Faddeev model in each sector, but a couple of important questions were left unanswered. We address those questions in this paper. Here is a summary of our description from [6].

Theorem 10 *Given a (smooth) map $\varphi : M \rightarrow S^2$, the primary homotopy invariant is the class $\varphi^* \mu_{S^2} \in H^2(M; \mathbb{Z})$. The map $\eta \mapsto (\varphi^* \mu_{S^2} \cup \eta)[M]$ from $H^1(M; \mathbb{Z})$ to \mathbb{Z} is a group homomorphism, and therefore has image $m\mathbb{Z}$ for some m determined by the class $\varphi^* \mu_{S^2}$. All maps $\psi : M \rightarrow S^2$ with the same second cohomology class $\psi^* \mu_{S^2} = \varphi^* \mu_{S^2}$ are obtained in the form*

$$\psi(x) = \Phi(x) \varphi(x) \Phi(x)^{-1},$$

where Φ is a map from M into S^3 . Furthermore, two maps $\Phi_1 \varphi \Phi_1^{-1}$ and $\Phi_2 \varphi \Phi_2^{-1}$ are homotopic if and only if $\deg \Phi_1 \equiv \deg \Phi_2 \pmod{2m}$.

For the special case of maps $S^3 \rightarrow S^2$ the primary homotopy invariant is trivial, so this theorem implies that every map $\varphi : S^3 \rightarrow S^2$ is related to the constant map $\mathbf{i} : S^3 \rightarrow S^2$ via a lift $\Phi : S^3 \rightarrow \text{Sp}(1)$ such that $\varphi(x) = \Phi^{-1}(x) \mathbf{i} \Phi(x) = \sigma \circ \Phi(x)$. This is exactly what one would expect from the interpretation of the primary invariant as a lifting obstruction. Notice that it is not the case that every map from a general 3-manifold to S^2 factors through the Hopf map in this way (the primary invariant is the obstruction to the existence of such a lift). This is why we had to introduce maps intertwining a pair of maps with the same primary invariant in our analytic description. The only homotopy invariant in the special case of maps $S^3 \rightarrow S^2$ is therefore the degree of the map Φ . As we reviewed at the start of this section such maps are classified up to homotopy by the Hopf invariant. Thus one would expect that $\text{Hopf}(\varphi) = -\deg \Phi$. This is indeed the case and the proof is given in Theorem 11 below.

The importance of the representation $\varphi(x) = \Phi^{-1}(x) \mathbf{i} \Phi(x)$ has been recognized since the work of Hurewicz [31]. That a continuous map $\varphi : S^3 \rightarrow S^2$ can be written as $\Phi^{-1} \mathbf{i} \Phi$ was proved, for example, in Pontrjagin's paper [39, Lemma 3]. We gave a different proof for the smooth case in our earlier paper on the existence of minimizers of the Faddeev functional [6]. R. Hardt and T. Rivière gave an analytic proof of the existence of the lift u in the case of a C^∞ map φ ([28, Lemma 2.1]; their notation is different from ours). They also noted that since $W^{1,3}(S^3, S^2)$ maps can be approximated by C^∞ maps, the Hopf number of such a map is well-defined either by approximation or as the degree of the corresponding $W^{1,3}$ lift, the latter approach has been used by T. Rivière in [40].

In their paper [34], dealing with the Faddeev energy functional, Fanghua Lin and Yisong Yang also invoke a ‘Hopf lift’ to address integrality of the Hopf number (note that they denote by u the map into S^2 and by \bar{u} its lift). The idea behind the integrality proof is quite simple and goes as follows. Assuming the existence of a lift Φ such that $\varphi = \Phi^{-1} \mathbf{i} \Phi$ one has $\varphi = \sigma \circ \Phi$, where σ is the Hopf map. This implies that $\varphi^* \omega_{S^2} = \Phi^* \sigma^* \omega_{S^2} = \Phi^* d\theta = d\Phi^* \theta$, where $\theta = \frac{1}{2\pi} \langle q^{-1} dq, q^{-1} \mathbf{i} q \rangle$. Thus we can take $\Phi^* \theta$ as the form in the definition of the Hopf invariant to compute

$$\text{Hopf}(\varphi) = \int_{S^3} \Phi^* \theta \wedge d\Phi^* \theta = \int_{S^3} \Phi^* (\theta \wedge d\theta) = - \int_{S^3} \Phi^* \omega_{S^3} = \deg \Phi.$$

This explanation extends easily to the maps $\varphi \in W^{1,3}$ since such maps can be approximated in the $W^{1,3}$ norm by smooth maps.

Adapting this argument for maps φ with finite Faddeev energy is more difficult. The problem is with the assertion that the degree of the lift is integral. F. Lin and Y. Yang consider maps from \mathbb{R}^3 into S^2 and S^3 . They refer to the paper [18] of M. Esteban and S. Müller for the integrality of the degree of the lift. In [18] the integral $\int_{\mathbb{R}^3} u^* \omega_{S^3}$ is proved to be an integer for the maps $u : \mathbb{R}^3 \rightarrow S^3$ with *finite Skyrme energy*, i.e., $u^{-1} du \in L^2$ and $u^{-1} du \wedge u^{-1} du \in L^2$. There is no justification in the paper of F. Lin and Y. Yang [34] that the lift \bar{u} they construct does have finite Skyrme energy. Indeed, Example 8 from Section 3 shows that lifts of finite Faddeev energy maps need not have finite Skyrme energy. The lift \bar{u} in [34] satisfies $\bar{u} \in \dot{W}^{1,2}(\mathbb{R}^3, S^3)$ and $\bar{u}^* \omega_{S^3} \in L^1(\mathbb{R}^3)$, but as one can see from the function from Example 6 in Subsection 2.4 this is not enough to justify integrality.

In Proposition 18, Section 3 we give an alternative construction of a lift that does have nice analytic properties, and we prove that $\text{Hopf}(\varphi) = \deg \Phi$ is an integer. We use these properties later when we consider the more complicated situation of maps from an arbitrary closed three-dimensional manifold M into S^2 , define the primary homotopy invariant, and compare pairs of such maps with the same primary invariant.

Our description of the homotopy classification together with the formula (Lemma 4) for the normalized volume form prompts the following definition of a numerical secondary homotopy invariant [7]. Given $\varphi, \psi : M \rightarrow S^2$ such that $\varphi^* \mu_{S^2} = \psi^* \mu_{S^2}$ define

$$\Upsilon(\varphi, \psi) = -\frac{1}{12\pi^2} \int_M \text{Re} (\Phi^{-1} d\Phi \wedge \Phi^{-1} d\Phi \wedge \Phi^{-1} d\Phi) \quad \text{mod} (2m_\psi), \quad (16)$$

where Φ is the map intertwining φ and ψ , i.e., $\psi = \Phi \varphi \Phi^{-1}$, and m_ψ is the divisibility of the class $\psi^* \mu_{S^2}$. The following lemma is used in the result first stated in [7] that relates the Hopf invariant and Υ in case the class $\psi^* \mu_{S^2}$ is torsion.

Theorem 11 *Let N be the smallest positive integer such that $N \varphi^* \mu_{S^2} = 0$. We have*

$$\text{Hopf}(\varphi) = \frac{1}{N^2} \Upsilon(\varphi_N, \mathbf{i}),$$

where φ_N is the composition of φ with the map $z \mapsto z^N$ of $S^2 = \mathbb{C} \cup \{\infty\}$, and \mathbf{i} is the constant map from M to $\mathbf{i} \in S^2$.

Proof. Define a map $p_N : \mathbb{C}P^1 \rightarrow \mathbb{C}P^1$ by $p_N([z : w]) = [z^N, w^N]$. Since $\mathbb{C}P^1$ is diffeomorphic to S^2 we may consider p_N as a smooth map of degree N on S^2 . This implies that there is a smooth 1-form α on S^2 such that $\frac{1}{N}p_N^*\omega_{S^2} = \omega_{S^2} + d\alpha$.

Given φ as in the proposition, the map $\varphi_N = p_N \circ \varphi$ has the same regularity as φ and satisfies $\varphi_N^*\mu_{S^2} = 0$. By Theorem 10 there is a map $\Phi : M \rightarrow S^3$ satisfying $\varphi_N = \Phi^{-1}\mathbf{i}\Phi$. Denote

$$\theta_N = \frac{1}{2\pi N} \langle \Phi^{-1}d\Phi, \varphi_N \rangle.$$

Equation (14) in Lemma 9 together with the quaternionic form of ω_{S^2} (Lemma 4) shows that $d\theta_N = \frac{1}{N}\varphi_N^*\omega_{S^2}$, which is equal to $\frac{1}{N}\varphi^*p_N^*\omega_{S^2} = \varphi^*\omega_{S^2} + d\varphi^*\alpha$. Thus

$$d\theta_N = \varphi^*\omega_{S^2} + d\varphi^*\alpha$$

and

$$\varphi^*\omega_{S^2} = d\theta, \quad \text{where} \quad \theta = \theta_N - \varphi^*\alpha.$$

Now,

$$\begin{aligned} \theta \wedge d\theta &= (\theta_N - \varphi^*\alpha) \wedge d(\theta_N - \varphi^*\alpha) \\ &= \theta_N \wedge d\theta_N + d(\theta_N \wedge \varphi^*\alpha) + \varphi^*\alpha \wedge d\varphi^*\alpha - 2\varphi^*\alpha \wedge d\theta_N. \end{aligned}$$

The terms $\varphi^*\alpha \wedge d\varphi^*\alpha - 2\varphi^*\alpha \wedge d\theta_N$ vanish, because they are the pull-backs of 3-forms on S^2 . Applying Lemma 7, formula (8), we obtain

$$\theta_N \wedge d\theta_N = -\frac{1}{N^2} \frac{1}{12\pi^2} \operatorname{Re}((\Phi^{-1}d\Phi)^3).$$

Finally,

$$\operatorname{Hopf}(\varphi) = \int_M \theta \wedge d\theta = -\frac{1}{N^2} \frac{1}{12\pi^2} \int_M \operatorname{Re}((\Phi^{-1}d\Phi)^3) = \frac{1}{N^2} \Upsilon(\varphi_N, \mathbf{i}).$$

□

The factor of N^2 in this last theorem can easily be understood from the linking number description of the Hopf invariant. Figure 1 displays the inverse image of a regular point under the Hopf map on the left (the inverse image of a second regular point would just be a parallel copy of the curve). The linking number between these two inverse images can be seen as the number of crossings in the figure. The inverse image of one regular point under the composition of a degree 3 self-map of S^2 with the Hopf invariant is drawn on the right. Clearly the linking number multiplies by 3^2 .

The open questions from [6] that we address in this paper are related to generalizing the description of the homotopy classification given in Theorem 10 to finite energy maps. Since finite energy maps may be discontinuous, the correct definition of $\varphi^*\mu_{S^2}$ is not clear. *A priori*, the integral Υ only takes integer values for smooth maps, so one would like to know that it takes integer values for finite-energy maps. We resolve these questions by defining $\varphi^*\mu_{S^2}$ for finite energy maps in Section 4 and proving that Υ only takes integer values in Section 5. We next present a number of examples of maps.



Figure 1: Change in Hopf invariant from composition with a degree 3-map

2.1.5 Examples

Maps from 3-manifolds to S^3 are classified by the degree, but it is instructive to construct specific representatives. For a geometer, the easiest way to construct a degree one map from an arbitrary 3-manifold M to S^3 is to take a cell decomposition of M with only one 3-cell and collapse the 2-skeleton $M^{(2)}$. Clearly, the inverse image of a generic point under this map consists of just one point. One can change the orientation to make the degree either $+1$ or -1 . Maps of any degree can then be obtained by postcomposition with self maps of S^3 of the right degree. The maps constructed in this way will be continuous but are unlikely to be smooth. Of course a well-known theorem of Whitehead tells us that we can approximate any continuous map arbitrarily closely by a smooth map in the same homotopy class.

Here is an alternate construction that produces smooth maps to begin with. The first step is to produce a map $\mathbb{R}^3 \rightarrow S^3$ which can be patched into any 3-manifold later. The map $\mathbb{R}^3 \rightarrow S^3$ is a modification of the stereographic projection taking the exterior of the unit ball to a pole. It is a composition of inversion ($x \mapsto |x|^{-2}x$), rescaling ($x \mapsto \rho(|x|)x$) via a cut-off function ρ equal to zero on $[0, 1]$ and equal to one on $[2, \infty)$, and stereographic projection ($x \mapsto (1 - |x|^2 + 2x)(1 + |x|^2)^{-1}$), i.e.,

$$x \mapsto \frac{1 - |\rho(|x|^{-1})|x|^{-2}x|^2 + 2\rho(|x|^{-1})|x|^{-2}x}{1 + |\rho(|x|^{-1})|x|^{-2}x|^2}.$$

Given any local chart in a 3-manifold M , i.e., a copy of \mathbb{R}^3 embedded in M , we can smoothly map M to S^3 by using this map on the copy of \mathbb{R}^3 and mapping the rest to the pole. This will be a smooth map of degree $+1$ or -1 depending on the orientation of the copy of \mathbb{R}^3 . Of course this is just a smoothing of the collapsing map first described.

Of course the same construction will give a degree one map from any closed n -manifold to S^n .

Given a finite collection of points in M one can take disjoint copies of \mathbb{R}^3 around each point and map each to S^3 with our standard map and map the rest of M to the pole to obtain a map of arbitrary degree. These maps can be viewed as collections of lumps representing particles and anti-particles. Moving the lumps or canceling/adding lump-antilump pairs does not change the homotopy class of the map. In this interpretation the degree is the net number of particles.

We now turn to the case of S^2 -valued maps. By the end of our examples we will have a picture of S^2 -valued maps similar to our picture of S^3 -valued maps. The Hopf map itself is the most important example of a map from a 3-manifold to S^2 . The Hopf map is just the result of conjugating the constant map by a degree one map from S^3 to S^3 . According to our general theory, one can obtain representatives of every homotopy class of maps by conjugating one map with each primary invariant by maps from M to S^3 .

Example 0. $M = S^3$. One can check that the map $q \mapsto q^n$ has degree n . It follows that every map from S^3 to S^2 is homotopic to exactly one of the form $q \mapsto q^{-n} \mathbf{i} q^n$.

Example 1. $M = S^2 \times S^1$. The second cohomology group of $S^2 \times S^1$ is the infinite cyclic group generated by $p^* \mu_{S^2}$ where $p : S^2 \times S^1 \rightarrow S^2$ is projection and μ_{S^2} is the generator of $H^2(S^2; \mathbb{Z})$. A map with primary invariant $m p^* \mu_{S^2}$ is constructed by composition of projection with a degree m self map of S^2 . Let p_m be such a map. Let $q : S^2 \times S^1 \rightarrow S^3$ be a degree one map. Every map $S^2 \times S^1 \rightarrow S^3$ is homotopic to exactly one of

$$q^{-n} p_m q^n \quad \text{for } n = 1, \dots, 2|m| \neq 0 \quad \text{or} \quad q^{-n} p_0 q^n.$$

Example 2. $M = T^3$, the 3-torus. We follow the same outline to construct maps here that we did in the $S^2 \times S^1$ case: we first construct maps with given primary invariant, and then twist these to obtain the rest.

Represent the 3-torus as $\mathbb{R}^3/\mathbb{Z}^3$. The second cohomology of T^3 with integer coefficients may then be viewed as the following subset of the second deRham cohomology, $\{(m_1, m_2, m_3) = m_1 dx^2 \wedge dx^3 + m_2 dx^3 \wedge dx^1 + m_3 dx^1 \wedge dx^2\}$ with m_k integers. Maps with given primary invariant can be constructed as the composition of a ‘linear’ projection from T^3 to T^2 followed by the collapse of the 1-skeleton taking T^2 to S^2 . Viewing T^2 as $\mathbb{R}^2/\mathbb{Z}^2$ an integer 2×3 matrix gives a well defined map from T^3 to T^2 . The induced map on the second cohomology is just given by the determinants of the 2×2 minors of the transpose of this matrix. It is an elementary exercise in number theory to see that any triple of integers arises as the determinants of the minors of such a matrix. For example,

$$\begin{bmatrix} 3 & 1 & 2 \\ -5 & 0 & -2 \end{bmatrix},$$

corresponds to the primary invariant $(-2, -4, 5)$. Let $p_{(m_1, m_2, m_3)}$ denote the map with primary invariant (m_1, m_2, m_3) . Let $q : T^3 \rightarrow S^3$ be a degree one map, then every map $T^3 \rightarrow S^2$ is homotopic to exactly one of

$$q^{-n} p_{(m_1, m_2, m_3)} q^n \quad \text{for } n = 1, \dots, 2 \gcd(m_1, m_2, m_3) \neq 0 \quad \text{or} \quad q^{-n} p_{(0,0,0)} q^n.$$

Example 3. $M = S^3/\mathbb{Z}_2$, the 3-dimensional projective space. The interesting thing about this case is that the primary invariant takes values in $H^2(S^3/\mathbb{Z}_2) \cong \mathbb{Z}_2$. Every possible primary invariant is taken by a map of the form $\varphi_k : S^3/\mathbb{Z}_2 \rightarrow S^2$, given by $\varphi_k([q]) := q^k \mathbf{i} q^{-k}$.

The primary invariant is trivial if and only if k is even and equal to the non-zero element of $H^2(S^3/\mathbb{Z}_2) \cong \mathbb{Z}_2$ when k is odd. We can see this because φ_0 is the constant map and each $\varphi_{2\ell}$ is related to φ_0 by conjugation by a map from $S^3/\mathbb{Z}^2 \rightarrow S^3$. Likewise all $\varphi_{2\ell+1}$ have the same primary invariant and one sees that the primary invariant of φ_1 is represented by the inverse image of a regular point which is exactly the geometric image of a great circle in S^3 in S^3/\mathbb{Z}_2 . This represents the nonzero element of $H_1(S^3/\mathbb{Z}_2)$.

To continue and obtain all homotopy classes of maps we just have to intertwine (conjugate) one of each primary invariant with maps from S^3/\mathbb{Z}_2 to S^3 having all possible degree. In this case one should notice that the map $[q] \mapsto q^2$ is well defined and has degree one since the inverse image of one point is plus and minus some point which represents just one point in S^3/\mathbb{Z}_2 . Thus φ_k represent all homotopy classes of maps.

There is a uniform way to describe all homotopy classes of maps from any 3-manifold to S^2 . Start with an embedded copy of $S^1 \times \mathbb{R}^2$ in the 3-manifold. Project any point in this solid torus to \mathbb{R}^2 then map \mathbb{R}^2 to S^2 by the map analogous to the map that we constructed from $\mathbb{R}^3 \rightarrow S^3$ at the start of the section. Map the rest of the 3-manifold to the pole. This is the Pontrjagin-Thom construction. In general any homotopy class of maps from a manifold to a sphere can be represented in this way. The primary invariant of a S^2 -valued map is the Poincaré dual of the core ($S^1 \times \{0\}$) of the solid torus. Roughly, the value of this 2-cohomology class on a 2-cycle is the number of points in the intersection of the cycle with the core of the solid torus. The secondary invariant measures the relative number of twists between two solid tori representing the same primary invariant. If an energy density is zero when the derivative is zero, the core of such a solid torus will be a vortex line (the inverse image of a regular point, a codimension 2 submanifold around which energy concentrates). Of course multiple vortex lines can be inserted into any 3-manifold. If the map evolves in a way preserving the homotopy class, the vortex lines will persist. More precisely the homology class generated by the vortex lines will be constant. There is some numerical evidence that minimizers of the Faddeev functional concentrate along such vortex lines, [21].

Returning to the examples from this section we can see the vortex line structure by taking the inverse image of a regular point. For Example 1 ($S^2 \times S^1$) the inverse image of a regular point under a degree m self map of S^2 is m points which then give m copies of the fiber $\{\text{point}\} \times S^1$ as vortex lines. As conjugation by a quaternion corresponds to rotation of \mathbb{R}^3 , conjugating these maps just ‘twists’ the vortices. Similarly, for Example 2 (T^3), the inverse image of a regular point just produces $\gcd(m_1, m_2, m_3)$ parallel copies of a circle in the (m_1, m_2, m_3) -direction.

2.2 Sobolev maps between manifolds

Over the last thirty years Sobolev maps between manifolds have become an indispensable part of the mathematical arsenal of topologists, geometers, analysts, and theoretical physicists. Many facts about Sobolev maps can be carried over from the well developed theory of Sobolev spaces on \mathbb{R}^n , [1]. However, there are interesting analytical problems specific to maps between manifolds that attract considerable attention, see [14] and references therein.

Not surprisingly the analytical complications arise when there are topological obstructions that are not respected by the spaces in question.

Recall, that for a domain Ω , s a non-negative integer and $p \geq 1$, a function f belongs to $W^{s,p}(\Omega)$ if it is Lebesgue measurable on Ω and f itself and all of its partial derivatives of order up to s are integrable with power p over Ω . The partial derivatives can be understood in terms of distributions. Identifying the functions that differ on locally negligible (measure 0) sets, and introducing the norm $\|f\|_{W^{s,p}} = \sum_{|\alpha| \leq s} \|\partial^\alpha f\|_{L^p(\Omega)}$, makes $W^{s,p}(\Omega)$ into a Banach space. This definition extends easily to vector-valued functions: an \mathbb{R}^N -valued function f belongs to $W^{s,p}(\Omega; \mathbb{R}^N)$ if each of its N components f^1 through f^N belongs to $W^{s,p}(\Omega)$. The norm is the sum of the norms of the components. Also recall that it is traditional in differential topology to define smooth functions on closed subsets as those that are restrictions of smooth functions from open subsets containing the given closed subset.

Another extension is to functions on a manifold. Let X be a smooth, closed, connected manifold of dimension n . Let $\{(U_i, \chi_i)\}$ be a finite atlas, and let $\{\vartheta_i\}$ be a partition of unity corresponding to the cover $\{U_i\}$. The Sobolev space $W^{s,p}(X, \mathbb{R}^N)$ is comprised of functions $f : X \rightarrow \mathbb{R}^N$ such that each product $f_i = \vartheta_i \cdot f$ is in $W^{s,p}(X, \mathbb{R}^N)$, i.e., the composition $f_i \circ \chi_i^{-1}$ is in $W^{s,p}(\mathbb{R}^n, \mathbb{R}^N)$. We set

$$\|f\|_{W^{s,p}(X, \mathbb{R}^N)} = \sum_i \|f_i \circ \chi_i^{-1}\|_{W^{s,p}(\mathbb{R}^n, \mathbb{R}^N)}.$$

When the cover $\{U_i\}$ and/or partition of unity $\{\vartheta_i\}$ are changed, the Sobolev spaces do not change, but the norms change to the equivalent ones. The space $W^{0,p}(X, \mathbb{R}^N)$ is identified with the Lebesgue space $L^p(X, \mathbb{R}^N)$.

There is a more traditional definition of Sobolev spaces on a manifold. Choose a smooth Riemannian metric on X . This, in particular, defines a measure, $d\text{vol}$, on X . The L^p spaces are defined with respect to this measure. The Sobolev space $W^{s,p}$, when s is a positive integer, can be defined using covariant derivatives as the closure of smooth functions in the norm $\sum_{\ell=0}^s \|\nabla^\ell f\|_{L^p}$. For non-integral s , the spaces $W^{s,p}$ are defined, usually, via real interpolation between spaces L^p and $W^{m,p}$, and are, in fact, Besov spaces, $B^{s;p,p}$, see [1]. For Sobolev spaces on Riemannian manifolds, see [4, 29].

Since we will be discussing and generalizing results from differential topology to Sobolev-valued differential forms, it is worth spelling out the relevant definitions here. The covariant derivative also acts on differential forms, so Sobolev spaces of forms $W^{s,p}(\wedge^k X)$ are just the completion of the space of smooth forms $C^\infty(\wedge^k X) = \Omega^k(X)$ with respect to the norm

$$\|\alpha\|_{W^{s,p}}^p := \sum_{m=0}^{\infty} \int_X |\nabla^m \alpha|^p d\text{vol}.$$

For such forms we say that $d\alpha = \beta$ (α a k -form) in the sense of distributions provided

$$\int_X \beta \wedge \gamma + (-1)^k \alpha \wedge d\gamma = 0,$$

for all compactly supported smooth $(n - k)$ -forms γ .

For the wedge product and pull-back we take the most concrete definition possible, namely a coordinate definition. There are fancier ways to address this issues used in geometric measure theory, but we will not need them in this paper. The coordinate definition of pull-back

$$\begin{aligned} & u^* (\alpha_{k_1, \dots, k_p}(x^1, \dots, x^n) dx^{k_1} \wedge \dots dx^{k_p}) \\ &= \sum_{\ell} \alpha_{k_1, \dots, k_p}(u^1(y), \dots, u^n(y)) \frac{\partial u^{k_1}}{\partial y^{\ell_1}} \dots \frac{\partial u^{k_p}}{\partial y^{\ell_p}} dy^{\ell_1} \wedge \dots dy^{\ell_p}, \end{aligned}$$

requires that the function u have sufficient regularity (but u need not be smooth) and one must prove that it is well-defined (independent of coordinate system) for sufficiently regular u . Similar comments hold for the coordinate definition of the wedge product.

To define manifold-valued Sobolev spaces, the target manifold, Y , is assumed to be isometrically embedded into some Euclidean space \mathbb{R}^N , and $W^{s,p}(X, Y)$ is understood as follows:

$$W^{s,p}(X, Y) = \{u \in W^{s,p}(X, \mathbb{R}^N) \mid u(x) \in Y \text{ for a.e. } x\}.$$

We will include several useful examples of functions belonging to some of these Sobolev spaces after we review the definition of the Faddeev and Skyrme functionals.

Over the past decade there has been a considerable progress made in understanding the topology of Sobolev, $W^{s,p}$, maps between manifolds. The list of publications on the subject is large and ever growing. An excellent overview of the main directions in this research is given by H. Brezis in [14]. A useful observation is that when the Sobolev space does respect the conventional topology, the conventional topological properties can be extended to the discontinuous Sobolev maps. When the Sobolev space does not respect the conventional topology, various bad things happen (for example, the homotopy classes of such maps degenerate: every map can be deformed to a constant). Of course, one should make clear what ‘respect’ means.

If we talk about homotopy invariants of smooth maps from M into N , and we have analytic expressions (of course, one should look for least restrictive analytic expressions) for all those invariants, and we see that the expressions make sense (can be computed) for *all* maps in some Sobolev space $W^{s,p}(M; N)$, then we say that $W^{s,p}(M; N)$ respects the (conventional) topology. Then we would expect that the numerical range of the invariants computed for maps in $W^{s,p}(M; N)$ is the same as for smooth maps (and attempt to prove this). Then we would expect that the homotopy classes for $W^{s,p}$ -maps each contains the corresponding class of smooth maps (and attempt to prove this). On the other hand, if the space $W^{s,p}(M; N)$ does not respect the topology, then this nice picture most probably breaks down, and we may study where it breaks down and to what extent. This general point of view has been and is guiding our work on the topology of Skyrme and Faddeev models. We bring in an additional point that it is important to consider other functionals defined on the maps $M \rightarrow N$, and interesting functionals come from physics.

2.3 The Faddeev and Skyrme functionals

The Skyrme functional was introduced in 1961 as an effective field theory for nucleon interactions. This model received renewed interest in 1983 when Witten described one Fermionic quantization of the model with structure group $SU(3)$ [51]. The functional was originally formulated for maps from \mathbb{R}^3 to $SU(2)$. The natural generalization defined on maps from a 3-manifold M to a Lie group G is

$$E(u) = \int_M |u^{-1}du|^2 + |u^{-1}du \wedge u^{-1}du|^2 d\text{vol}_M.$$

The original Faddeev functional was defined for maps from \mathbb{R}^3 to S^2 . It extends naturally to maps $\varphi : M \rightarrow S^2$, [19].

$$E(\varphi) = \int_M |d\varphi|^2 + |d\varphi \wedge d\varphi|^2 d\text{vol}_M. \quad (17)$$

Just to be clear about what is meant here, writing $\varphi = \varphi^1 \mathbf{i} + \varphi^2 \mathbf{j} + \varphi^3 \mathbf{k}$, we have

$$\begin{aligned} |d\varphi|^2 &= |d\varphi^1|^2 + |d\varphi^2|^2 + |d\varphi^3|^2, \\ |d\varphi \wedge d\varphi|^2 &= |d\varphi^1 \wedge d\varphi^2|^2 + |d\varphi^2 \wedge d\varphi^3|^2 + |d\varphi^3 \wedge d\varphi^1|^2. \end{aligned}$$

The lowest regularity of admissible maps for the Faddeev functional is a little bit better than $W^{1,2}(M; S^2)$, for the integral of the quartic term to be finite.

Analysis of Sobolev maps is the analysis of the functional on the space of maps that corresponds to the Sobolev norm. Some of the Sobolev norms preserve homotopy information, some do not. One of the points we would like to make is that there are functionals other than Sobolev norms, that may be as interesting or even more interesting from the viewpoint of the topology of manifolds or from the viewpoint of Physics. The Skyrme and Faddeev functionals are two such examples. Of course preserving homotopy is desirable from the viewpoint of topology, see [9] for the reasons why these functionals are ‘natural’ from the viewpoint of Physics.

2.4 Sample functions

Good functions to keep in mind when thinking about Sobolev spaces are radial functions with variable rates of growth and oscillation. Let D^n denote an open unit disk (ball) in \mathbb{R}^n centered at the origin. The function $f : D^n \rightarrow \mathbb{R}$ given by $f(x) = |x|^{-n/q}$ is in $W^{s,p}(D^n)$ with $s > 0$ if $|x|^{-(n/q+s)}$ is in L^p , which means $(n/q + s)p - (n - 1) < 1$, i.e., $n/q + s < n/p$. The function $|x|^{-n/q}$ does not belong to $L^q(D^n)$. Note that, by the Sobolev embedding theorem, the space $W^{s,p}(D^n)$ with $s > 0$ embeds continuously into $L^q(D^n)$ when the indices satisfy $1 \leq p, q$ and $s - n/p \geq -n/q$.

Example 4. Arguably the most useful map from D^k to \mathbb{R}^k is the map $\phi : x \mapsto x/|x|$. This map is bounded and its derivative behaves like $1/|x|$ which is in $L^p(D^k)$ provided

$p < k$. Thus, $\phi \in W^{1,p}(D^k, \mathbb{R}^k)$ for $1 \leq p < k$. This map can be generalized to higher dimensions as a map $\phi : D^k \times D^{n-k} \rightarrow \mathbb{R}^n$ given by $\phi(x, y) = (x/|x|, y)$. It follows that $\phi \in W^{1,p}(D^k \times D^{n-k}, \mathbb{R}^n)$ if and only if $p < k$. We conclude that there is no reasonable sense in which a $W^{1,p}$ map will preserve homotopy information on the k -skeleton for $p < k$. This illustrates the result of White [49]: The homotopy classes of maps with one derivative in L^p restricted to the k -skeleton of a manifold agree with the homotopy classes of smooth maps provided p is larger than k . In light of this, when M is a 3-manifold a ‘natural’ Sobolev space of maps $M \rightarrow S^2$ to consider is $W^{1,3}(M; S^2)$. This is because the degree and many other homotopy invariants can be defined for such functions. Brezis and Nirenberg [16] showed that the degree can be defined for functions belonging to VMO – the space of functions of *vanishing mean oscillation*. This is a larger space than $W^{1,3}(M; S^2)$. By definition a function f is in VMO if

$$\lim_{\epsilon \rightarrow 0} \sup_{D_R, 0 < R < \epsilon} |D_R|^{-1} \int_{D_R} |f - |D_R|^{-1} \int_{D_R} f| = 0,$$

where D_R is a disk of radius R . Functions in VMO (and thus $W^{1,3}(M; S^2)$) can be approximated arbitrarily closely in the BMO norm by smooth functions, see [16]. As we show in this paper the functions with finite Faddeev energy are also ‘natural’ from the point of view of topology. There are finite Faddeev energy functions that are not in VMO, so arguments in the finite energy case are more subtle; see Example 5 below.

Several maps similar to the ones in Example 4 may be used to clarify the distinction between $W^{1,p}$ maps and finite Faddeev or Skyrme energy maps. The map $\eta_1 : D^3 \rightarrow S^2 \subset \mathbb{R}^3$ given by $\eta_1(x) = x/|x|$ is in $W^{1,p}$ for $p < 3$, but is not in $W^{1,3}$ and does not have finite Faddeev energy. The composition of the projection of S^3 to D^3 with this map η_1 can be patched into any smooth map from a 3-manifold to obtain a similar example. It follows using our observation after Example 4 that our results extending homotopy invariants to finite Faddeev energy or $W^{1,3}$ maps are sharp.

Example 5. The function, $\eta_2 : D^3 \rightarrow S^2$ given by $\eta_2(x) = \cos(\ln |x|)\mathbf{i} + \sin(\ln |x|)\mathbf{j}$ is in $W^{1,2}$ and has finite Faddeev energy but is not in $W^{1,3}$, and the function $\eta_3 : D^3 \rightarrow S^1$ given by $\eta_3(x) = \cos(\ln |\ln(|x|)|)\mathbf{i} + \sin(\ln |\ln(|x|)|)\mathbf{j}$ is in $W^{1,3}$ but is not continuous. These last two functions may be patched into maps from an arbitrary 3-manifold. Furthermore they may be composed with maps into S^3 or any non-trivial compact Lie group. These last examples show that the finite energy maps need not be continuous, so a special care is needed when defining homotopy invariants for maps in these spaces. In fact, Diego Maldonado pointed out that the function η_2 is not even in VMO. To see this recall that a vector-valued function is in VMO exactly when each component is in VMO and notice that

$$\int r^2 \cos(\ln r) dr = \frac{3}{10} r^3 \cos(\ln r) + \frac{1}{10} r^3 \sin(\ln r) + C,$$

so the integral in the definition of VMO can be evaluated explicitly for disks centered at the origin.

The following example is a variant of Example 4 that illustrates certain issues that we resolve in this paper.

Example 6. Let ϑ be coordinates on S^{n-1} , so that generalized polar coordinates on \mathbb{R}^n may be written (r, ϑ) . Using these coordinates for \mathbb{R}^n and generalized cylindrical coordinates (r, ϑ, z) for \mathbb{R}^{n+1} , the stereographic parametrization of S^n may be expressed as $(r, \vartheta) \mapsto (\frac{2r}{r^2+1}, \vartheta, \frac{r^2-1}{r^2+1})$. In this parametrization the volume form on S^n may be written as

$$d\text{vol}_{S^n} = 2^n(r^2 + 1)^{-n} d\text{vol}_{\mathbb{R}^n} = 2^n r^{n-1} (r^2 + 1)^{-n} dr d\text{vol}_{S^{n-1}}.$$

Consider the map $\Phi : S^n \rightarrow S^n$ given in terms of coordinates in \mathbb{R}^n by

$$\Phi(r, \vartheta) = \begin{cases} (r, \vartheta) & \text{if } r \leq 1 \\ (1, \vartheta) & \text{if } r \geq 1 \end{cases}$$

Geometrically the map Φ is the identity on the lower hemisphere glued to the ubiquitous map $x \mapsto x/|x|$ on the upper hemisphere (Φ projects points in the upper hemisphere to the point on the equator meeting the great circle passing through the given point and the north pole). The derivatives of Φ are bounded on the lower hemisphere and grow as r on the upper hemisphere. For the integral

$$\int_0^\infty \frac{r^{n-1}}{(1+r^2)^n} r^p dr$$

to be finite, one must have $p < n$. This implies $\Phi \in W^{1,p}(S^n, S^n)$ for $p < n$. This is slightly less regularity than what is required for a well-defined degree. However, the integral expression for the degree of Φ does make sense and yields $\int_{S^n} \Phi^* \omega_{S^n} = \frac{1}{2}$. This example can easily be modified to show that every real number is the ‘degree’ of a map in $W^{1,p}(S^n, S^n)$ when $p < n$.

Example 7. The usual identities from differential topology no longer hold when the regularity of the functions in question is too low. In particular, the pull-back and the differential may not commute. Consider the ubiquitous map $u; z \mapsto z/|z|$ from D^2 to \mathbb{R}^2 . If (t, ϕ) and (r, θ) are polar coordinates on the domain and the target respectively, we have $u(t, \phi) = (1, \phi)$. This is clearly in $W^{1,p}$ for every $p < 2$. Set $\alpha = r^2 d\theta$, so that $d\alpha = 2r dr \wedge d\theta$. It follows from the definition that $u^*(d\alpha) = 0$. On the other hand $u^*\alpha = d\phi$ and $d(u^*\alpha)$ is a multiple of the delta-function:

$$(f, d(u^*\alpha)) = - \int_{D^2} df \wedge u^*\alpha = - \lim_{\epsilon \rightarrow 0} \int_{D^2 \setminus \{|x| < \epsilon\}} df \wedge d\phi = \lim_{\epsilon \rightarrow 0} \int_{|x|=\epsilon} f d\phi = 2\pi f(0),$$

for any test function f in D^2 . This example generalizes to show that the pull-back with respect to the maps from Example 4 does not commute with exterior differentiation. As we have remarked, there is such a map in $W^{1,p}(D^n, D^n)$ for every $p < n$.

The maps whose pull-backs commute with the differentials of forms of certain degrees have special regularity properties and have been studied, see for example [25, Chapter I.3]. In our present paper this property has a prominent role. We need the intertwining maps $u : M^3 \rightarrow S^3$ to have it in order to prove that their degree is an integer. Starting from Proposition 18 we prove this property again and again for various maps. We cannot apply the machinery developed in [25] because we work not in the Sobolev but rather in finite energy environment. We develop our own technique to handle this situation.

3 Local representations

Two local representation theorems are key technical tools used in this paper. The first is the following result about the local structure of flat connections that we proved in [5, Lemma 3] and called the nonlinear Poincaré lemma. Let G be a compact Lie group and \mathfrak{g} be its algebra Lie. Let I^m denote a unit cube in \mathbb{R}^m .

Lemma 12 *Given any L^2 \mathfrak{g} -valued 1-form A on I^3 such that*

$$dA + \frac{1}{2}[A, A] = 0 \quad (18)$$

in the sense of distributions, there exists $u \in W^{1,2}(I^3, G)$ such that $u^{-1} \in W^{1,2}(I^3, G)$ and $A = u^{-1} du$. Furthermore, for any two such maps, u and v , there exists $g \in G$ so that $u(x) = g \cdot v(x)$, for almost every $x \in I^3$. If $A \in W^{k,2}$ then we can take $u \in W^{k+1,2}$. If $A \in C^\infty$ then we can take $u \in C^\infty$.

Proof. The case $A \in L^2$ is proved in [5, Lemma 3]. If we know that $A \in W^{1,2}$, the map u constructed in [5, Lemma 3] will belong to $W^{2,2}$. Indeed, differentiating $\partial_j u = u A_j$, we obtain $\partial_k \partial_j u = u A_k A_j + u \partial_k A_j$. Because u is bounded as a map into a compact group, the norm $\|u A_k A_j\|_{L^2}$ is less than a constant times $\|A\|_{L^4}^2$, which is finite due to Sobolev embedding $W^{1,2} \subset L^4$ in dimension 3. If A has derivatives of order 2 or higher in L^2 , then a similar argument shows that u has one derivative more than A in L^2 . \square

Remark 13 *The non-linear Poincaré lemma also holds for flat connections defined on any domain that is bilipschitz equivalent to the unit cube. Assuming that Ω is a domain bilipschitz equivalent to a cube and A is a flat connection in $L^2(\Omega)$, we can pull-back A to the cube and the result will still be flat (as one can see by the change of variables formula), so we can apply the non-linear Poincaré lemma to obtain the local developing map in $W^{1,2}(I^m)$ and change coordinates to obtain the desired local developing map on the domain. The change of variables formula implies that $f \circ g$ is in $W^{1,p}(X)$ exactly when f is in $W^{1,p}(Y)$ provided $p \geq 1$ and $g : X \rightarrow Y$ is bilipschitz.*

Remark 14 *In the historical development of differential topology the existence and uniqueness of piecewise linear structures on smooth manifolds was a fundamental question. In order to interpolate between the smooth and piecewise linear categories, the notion of piecewise smooth functions was introduced. Recall that an abstract simplicial complex K is a collection of finite non-empty subsets of a fixed vertex set such that any non-empty subset of an element of the collection is in the collection and the union of the entire collection is the vertex set. There is a standard construction of a geometric realization of an abstract simplicial complex (denoted by $|K|$) obtained by associating one standard basis vector for each vertex and taking the union of the convex hulls of the points associated to each set in the collection. The convex hull of the points associated to one of the sets in the collection is called a geometric simplex. A map from the geometric realization of an abstract complex into a manifold is called piecewise smooth if the restriction of the map to each geometric simplex is equal to the restriction*

of a smooth map from an open set containing the simplex. The closed star of any point in the geometric complex (denoted by $st(p)$) is the union of all geometric simplices containing that point. One defines a differential of a piecewise smooth map $f : |K| \rightarrow M$ at a point as the map $df_p : st(p) \rightarrow T_{f(p)}M$ given by $df_p(v) := \frac{d}{dt}f(tv + (1-t)p)|_{t=0}$. A map from a geometric complex to a subset of a manifold is a piecewise smooth equivalence if it is piecewise smooth, bijective with bijective differentials. It is a classical fact due to Whitehead that every smooth manifold is piecewise smoothly equivalent to some geometric complex; see [37]. In addition the image of any intersection of closed stars of vertices is piecewise smoothly equivalent to the unit cube. We use these facts when we need to decompose our manifold into reasonable domains later. Notice that piecewise smooth equivalence implies bilipschitz equivalence. From here forward we will use Ω to denote a domain piecewise smoothly equivalent to a cube.

We now describe the second local representation theorem. This theorem states that any sufficiently regular map $\varphi : \Omega \rightarrow S^2$ may be written in the form $\varphi = u^{-1}\mathbf{i}u$ for some lift $u : \Omega \rightarrow S^3$. Notice that this result must be local because there can be no lift $u : M \rightarrow S^3$ of a map $\varphi : M \rightarrow S^2$ when $\varphi^*\mu_{S^2} \neq 0$ because $H^2(S^3) = 0$. Also notice that when a lift does exist, there should be many lifts because S^3 has one more dimension than S^2 . In fact there is an infinite-dimensional group of gauge symmetries that act on the possible lifts. One may impose a local gauge-fixing condition to remove this ambiguity. As usual requiring the slice to be perpendicular to the orbit of the gauge group is a reasonable choice for gauge-fixing condition. This is the interpretation of the gauge fixing condition (22) that we include in our local representation theorem.

3.1 Finite energy pull-back of the area form

Before we get to the proof of Proposition 18 we would like to point out once again the main technical difficulties one encounters when generalizing usual results from differential topology to functions with low regularity/summability.

- The pull-back and the differential do not necessarily commute (Example 7 from Section 2.4). To establish $df^*\alpha = f^*d\alpha$ we need to fully exploit the structure of the form α and special properties (e.g. boundedness) of the map f . Where naïve mollification does not work, we decompose the form and approximate appropriate parts in a way preserving cancellations, apply the formula and take limits.
- The product formula $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta$ does not necessarily hold if α and/or β lack summability. Again, we use the structure of the forms to show that $\alpha \wedge \beta$ and each term on the right is in L^1_{loc} .

We need to establish an auxiliary result before considering the second local representation theorem. Namely, the pull-back of the normalized volume form ω_{S^2} by a finite Faddeev energy map φ is a closed 2-form on Ω .

Remark 15 In their paper [34], Fanghua Lin and Yisong Yang include the requirement of the (distributional) closedness of the pull-back in the definition of their class X_2 of maps where minimizers of the Faddeev energy are sought. Our lemma below shows that finite energy implies $d\varphi^*\omega_{S^2} = 0$.

Lemma 16 Let Ω be a domain piecewise smoothly equivalent to the cube. If φ is a finite Faddeev energy map in $W^{1,2}(\Omega, S^2)$, then the pull-back, $\varphi^*\omega_{S^2}$, is closed: $d\varphi^*\omega_{S^2} = 0$ in the sense of distributions. In particular,

$$d\varphi \wedge d\varphi \wedge d\varphi = 0. \quad (19)$$

Remark 17 The same result holds for finite Faddeev energy maps on \mathbb{R}^3 . Indeed to check that $\langle \varphi^*\omega_{S^2}, d\phi \rangle = 0$ for a smooth test form ϕ it is sufficient to check it on the closure of the support of ϕ . This is certainly contained in some cube and the restriction of φ to this large cube satisfies the hypothesis of the theorem.

Proof. Recall from Lemma 4 that,

$$\varphi^*\omega_{S^2} = -\frac{1}{8\pi} \varphi d\varphi \wedge d\varphi;$$

in particular the form $\varphi d\varphi \wedge d\varphi$ is real-valued. Consider $y^{-1}dy$ on $\mathbb{H} - \{0\}$ and set $A = \varphi^*(y^{-1}dy) = -\varphi d\varphi$. By mollification we can find a sequence of functions $\varphi_n : \Omega \rightarrow \mathbb{H}$ such that $\varphi_n \rightarrow \varphi$ in $W^{1,2}$. Applying the product rule to $\varphi_n d\varphi_n$ and passing to the limit we see that

$$dA = -A \wedge A = -d\varphi \wedge d\varphi$$

in the sense of distributions. Thus,

$$-\varphi d\varphi \wedge d\varphi = \varphi dA.$$

Since φ is in $W^{1,2}(\Omega, S^2)$ and has finite Faddeev energy, we conclude that $A \in L^2$ and $dA \in L^2$. Let T_ϵ denote a mollification operator. Then

$$d(\varphi_n dT_\epsilon(A)) = d\varphi_n \wedge dT_\epsilon(A) = d\varphi_n \wedge T_\epsilon(dA),$$

which converges to $d\varphi \wedge dA$ in L^1 . Thus, in the sense of distributions,

$$d\varphi^*\omega_{S^2} = -\frac{1}{8\pi} d\varphi \wedge d\varphi \wedge d\varphi.$$

Now, check that

$$d\varphi = -\frac{1}{2} [A, \varphi]$$

and compute

$$\begin{aligned}
8 d\varphi \wedge d\varphi \wedge d\varphi &= - [A, \varphi] \wedge [A, \varphi] \wedge [A, \varphi] \\
&= - \operatorname{Re} (A\varphi \wedge A\varphi \wedge A\varphi + A\varphi \wedge A \wedge A \\
&\quad - \varphi A \wedge A\varphi \wedge A\varphi - \varphi A \wedge A \wedge A \\
&\quad + A \wedge A \wedge A\varphi - A \wedge A \wedge \varphi A \\
&\quad + \varphi A \wedge \varphi A \wedge A\varphi - \varphi A \wedge \varphi A \wedge \varphi A) .
\end{aligned} \tag{20}$$

Using the fact that factors can be cycled under the real part we see that all terms happily cancel out. \square

3.2 Representation of S^2 -valued maps on cubes

We are now in a position to prove the local representation theorem for S^2 -valued maps.

Proposition 18 *Let Ω be a bounded region in \mathbb{R}^3 piecewise smoothly equivalent to the cube. If $\varphi : \Omega \rightarrow S^2$ is a finite Faddeev energy map, then there is a map $u \in W^{1,2}(\Omega, S^3)$ so that $\varphi = u^{-1} \mathbf{i} u$. In addition, $u^{-1} du \wedge u^{-1} du \wedge u^{-1} du \in L^{3/2}(\Omega)$, and u has the following important property: For any smooth \mathbb{R}^3 -valued function f on S^3 ,*

$$u^* (d\langle f, y^{-1} dy \wedge y^{-1} dy \rangle) = d (u^* (\langle f, y^{-1} dy \wedge y^{-1} dy \rangle)) . \tag{21}$$

Here $y^{-1} dy$ is the usual Maurer-Cartan form on S^3 . Furthermore, for any two such maps u and v there is a map λ in $W^{1,2}(\Omega, S^1)$ so that $v = \lambda u$. Given a smooth metric on Ω one can choose u as above with

$$\delta \langle u^{-1} du, \varphi \rangle = 0, \quad \text{and} \quad i^* * (\langle u^{-1} du, \varphi \rangle) = 0, \tag{22}$$

where δ is the codifferential defined via the metric, $*$ is the Hodge star operator and $i : \partial\Omega \rightarrow \Omega$ is the inclusion. Such a lift is unique up to left multiplication by a unit complex number. A similar result also holds for maps in $\varphi \in W^{1,3}(\Omega, S^2)$. In this case $u^{-1} du \wedge u^{-1} du \wedge u^{-1} du \in L^1$, and (21) holds.

Remark 19 *That $u \in W^{1,2}$, $u^{-1} du \wedge u^{-1} du \wedge u^{-1} du \in L^1$, and equality (21) holds imply that the map u is cartesian, $u \in \operatorname{cart}^1(\Omega, \mathbb{R}^4)$ in the sense of [25, Section I.3.2].*

Proof. Assume for a moment that we knew that such a map u existed. Then a computation starting with the derivative of $\varphi = u^{-1} \mathbf{i} u$ leads to $\varphi^{-1} d\varphi = a + \varphi a \varphi$ where $a = u^{-1} du$. Since $\varphi^{-1} d\varphi$ is perpendicular to φ , we can solve this last equation for a by splitting it into directions parallel and perpendicular to φ . We see that this equation holds exactly when $a = \frac{1}{2} \varphi^{-1} d\varphi + \varphi \xi$ for some real valued 1-form ξ . In fact ξ is just the φ component of a . Since a is flat we would have $0 = da + a \wedge a = \varphi d\xi - \frac{1}{4} d\varphi \wedge d\varphi$ or equivalently $d\xi = -\frac{1}{4} \varphi d\varphi \wedge d\varphi$. We will turn this around by solving for ξ then a and finally u .

We already saw in the proof of Lemma 4 that the form $\varphi d\varphi \wedge d\varphi$ is real. That it is closed follows from Lemma 16. When $\varphi \in W^{1,3}$ this form is in $L^{3/2}$ and when φ has finite Faddeev energy this form is in L^2 . It follows from the usual Poincaré lemma that there is a ξ in $W^{1,3/2}$ or $W^{1,2}$ respectively so that $d\xi = -\frac{1}{4}\varphi d\varphi \wedge d\varphi$, [35]. In fact there is a unique such form satisfying

$$d\xi = -\frac{1}{4}\varphi d\varphi \wedge d\varphi, \quad \delta\xi = 0, \quad \text{in } \Omega, \quad i^* * (\xi) = 0. \quad (23)$$

Set $a = \frac{1}{2}\varphi^{-1}d\varphi + \varphi\xi$ and observe that the Sobolev embedding theorem implies that a is in L^2 in either case. Now, taking into account (23) and that pointwise φ is purely imaginary and $\varphi^2 = -1$, we obtain

$$da = -\frac{1}{2}\varphi^{-1}d\varphi \wedge \varphi^{-1}d\varphi + d\varphi \wedge \xi + \varphi d\xi = -\frac{1}{4}d\varphi \wedge d\varphi + d\varphi \wedge \xi.$$

For the same reasons,

$$a \wedge a = \frac{1}{4}\varphi^{-1}d\varphi \wedge \varphi^{-1}d\varphi + \frac{1}{2}\varphi^{-1}d\varphi \wedge \varphi\xi + \frac{1}{2}\varphi\xi \wedge \varphi^{-1}d\varphi = \frac{1}{4}d\varphi \wedge d\varphi - d\varphi \wedge \xi.$$

Thus, $da + a \wedge a = 0$, i.e., a is flat. The nonlinear Poincaré lemma (Lemma 12) now implies the existence of a $w \in W^{1,2}(\Omega; S^3)$ with $a = w^{-1}dw$.

Compute: $(w\varphi w^{-1})^{-1}d(w\varphi w^{-1}) = w(\varphi^{-1}d\varphi - A - \varphi A\varphi)w^{-1} = 0$. Since $\varphi(x) \in S^2$ for almost all x , the product $w\varphi w^{-1}$ takes values in S^2 almost everywhere as well. Since its derivative vanishes, we conclude that there is a $y \in S^2$ and thus a unit quaternion $p \in S^3$ so that $w(x)\varphi(x)w(x)^{-1} = y = p^{-1}\mathbf{i}p$ almost everywhere. Clearly $u = pw \in W^{1,2}$ satisfies $\varphi = u^{-1}\mathbf{i}u$. When $\varphi \in W^{1,3}$ we have $u^{-1}du = a \in L^3$, thus $\text{Re}(u^{-1}du \wedge u^{-1}du \wedge u^{-1}du) \in L^1$.

Now assume that we only know that $\varphi \in W^{1,2}$ and has finite Faddeev energy. Recall that

$$u^{-1}du = a = \frac{1}{2}\varphi^{-1}d\varphi + \varphi\xi,$$

where ξ is a solution of the elliptic problem (23). Clearly, $\xi = \langle u^{-1}du, \varphi \rangle$, and so the gauge fixing conditions (22) are satisfied. As we have seen,

$$a \wedge a = \frac{1}{4}d\varphi \wedge d\varphi - d\varphi \wedge \xi.$$

Since φ is assumed to have finite Faddeev energy, $d\varphi \in L^2$ and $d\varphi \wedge d\varphi \in L^2$. Also, the solution of problem (23) satisfies $\xi \in W^{1,2} \subset L^6$. Thus, $a \wedge a \in L^{3/2}$. By the isoperimetric inequality (for a parallelepiped in \mathbb{R}^3)

$$|a \wedge a \wedge a| \leq C |a \wedge a|^{3/2},$$

thus $a \wedge a \wedge a$ is in L^1 . We will see a second proof establishing that $a \wedge a \wedge a \in L^{3/2}$, independently of the isoperimetric inequality later.

Given two maps u and v with $u^{-1}\mathbf{i}u = v^{-1}\mathbf{i}v$ set $\lambda = vu^{-1}$ and notice that $vu^{-1}\mathbf{i} = \mathbf{i}vu^{-1}$ implies that λ takes values in S^1 hence lives in $W^{1,2}(\Omega, S^1)$. We can write λ in the form $\lambda(x) = \exp 2\pi\mathbf{i}\theta(x)$ for some real-valued function $\theta \in W^{1,2}(U, \mathbb{R})$. This follows from [12, Lemma 1], or from our non-linear Poincaré Lemma 12 after noticing that $\lambda^{-1}d\lambda$ is a flat, $\mathbf{i}\mathbb{R}$ -valued connection. In any case, $v^{-1}dv = u^{-1}du + \varphi 2\pi d\theta$, and assuming that both u and v satisfy the gauge fixing condition (22) implies that $\delta d\theta = 0$, so we must have $d\theta = 0$, i.e., $\theta(x) = \text{const.}$

It remains to prove formula (21). We have

$$u^* (\langle f, y^{-1}dy \wedge y^{-1}dy \rangle) = \langle u^*f, a \wedge a \rangle.$$

Thus our goal is to prove

$$d(\langle u^*f, a \wedge a \rangle) = u^*d\langle f, y^{-1}dy \wedge y^{-1}dy \rangle. \quad (24)$$

When $u \in W^{1,3}$ this is a straight-forward application of the approximation argument. However when u is only assumed to have finite energy the argument is not obvious at all. The difficulty here, at the surface of it, is that $a \wedge a$ may not be in L^2 , and the best we can say about u^*df is that it is in L^2 . This means that it is hard to see that the approximations converge in L^1 to the right-hand side to make the approximation argument work. Thus, the only hope is to look at the structure of these expressions and find the right cancelations to get the proper convergence.

We start by re-writing the right-hand side of (24). To shorten the formulas it is convenient to adopt the summation over the repeated indices rule and to introduce the following temporary notation.

Set $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$, and $\mathbf{e}_3 = \mathbf{k}$. Let θ denote the Maurer-Cartan form $y^{-1}dy$ and write

$$y^{-1}dy = \theta = \theta^i \mathbf{e}_i.$$

The real 1-forms θ^1 , θ^2 , and θ^3 form an orthonormal basis in T^*S^3 . Denote by X_1 , X_2 , X_3 the dual basis of TS^3 , $\theta^m(X_\ell) = \delta_{\ell m}$. One easily checks that, for any function f ,

$$df = X_k(f)\theta^k. \quad (25)$$

We also have

$$\theta \wedge \theta = 2 (\theta^2 \wedge \theta^3 \mathbf{e}_1 + \theta^3 \wedge \theta^1 \mathbf{e}_2 + \theta^1 \wedge \theta^2 \mathbf{e}_3)$$

and

$$\langle \theta, \theta \wedge \theta \rangle = -\theta \wedge \theta \wedge \theta = 6 \theta^1 \wedge \theta^2 \wedge \theta^3.$$

Writing an \mathbb{R}^3 -valued (i.e., purely imaginary) function f as $f = f^i \mathbf{e}_i$, we have

$$d\langle f, \theta \wedge \theta \rangle = \frac{1}{3} X_m(f^m) \langle \theta, \theta \wedge \theta \rangle.$$

We used the fact that terms such as $\langle X_2(f^1)\theta^2\mathbf{e}_1, \theta \wedge \theta \rangle = 2 X_2(f^1)\theta^2 \wedge \theta^2 \wedge \theta^3$ vanish. Since

$$u^*\theta = u^{-1}du = a,$$

the right-hand side of our goal (24) is

$$u^*(d\langle f, \theta \wedge \theta \rangle) = \frac{1}{3} u^*(X_m(f^m)) \langle a, a \wedge a \rangle. \quad (26)$$

Turning to the left-hand side of (24), recall that

$$a = -\frac{1}{2} \varphi d\varphi + \varphi \xi$$

with $\varphi, \xi \in W^{1,2}$. To see and exploit cancellations, we use a special approximation a_n to a ; namely,

$$a_n := -\frac{1}{2} \varphi d\varphi + \varphi \xi_n,$$

where ξ_n is a sequence of smooth real 1-forms converging to ξ in $W^{1,2}$ (and hence in L^6). This gives

$$a_n \wedge a_n = \frac{1}{4} d\varphi \wedge d\varphi - d\varphi \wedge \xi_n$$

and one sees that $a_n \wedge a_n \rightarrow a \wedge a$ in $L^{3/2}$. It follows that $\langle u^*f, a_n \wedge a_n \rangle \rightarrow \langle u^*f, a \wedge a \rangle$ in $L^{3/2}$ (since f is bounded), so

$$d\langle u^*f, a_n \wedge a_n \rangle \rightarrow d\langle u^*f, a \wedge a \rangle$$

as distributions. We will see that this sequence also converges to $u^*d\langle f, y^{-1}dy \wedge y^{-1}dy \rangle$ to establish our goal (24).

Note that $a_n \wedge a_n$ is in L^2 since φ has finite Faddeev energy. Also, $d(a_n \wedge a_n) = d\varphi \wedge d\xi_n$ in the sense of distributions. Moreover, $d\varphi \wedge d\xi_n \rightarrow d\varphi \wedge d\xi$ in L^1 , and $d\varphi \wedge d\xi = \frac{1}{4} \varphi d\varphi \wedge d\varphi = 0$ by Lemma 16. Since u^*f is bounded with derivative in L^2 , we have

$$d\langle u^*f, a_n \wedge a_n \rangle = \langle d(u^*f), a_n \wedge a_n \rangle + \langle u^*f, d\varphi \wedge d\xi_n \rangle.$$

As $n \rightarrow \infty$, the second term on the right goes to 0. Let us analyze the first term.

Since u is in $W^{1,2}$ and f is smooth, $d(u^*f) = u^*(df)$. Applying formula (25) to $f = f^j \mathbf{e}_j$, we get

$$d(u^*f) = u^*(X_i(f^j))u^*(\theta^i)\mathbf{e}_j.$$

Noting that $\theta^i = \langle \theta, \mathbf{e}_i \rangle$ and $a = u^*\theta$, we write

$$u^*(\theta^m) = \langle a, \mathbf{e}_m \rangle = \langle a - a_n, \mathbf{e}_m \rangle + \langle a_n, \mathbf{e}_m \rangle.$$

Then,

$$\langle d(u^*f), a_n \wedge a_n \rangle = u^*(X_i(f^j))\langle a - a_n, \mathbf{e}_i \rangle \langle a_n \wedge a_n, \mathbf{e}_j \rangle + u^*(X_i(f^j))\langle a_n, \mathbf{e}_i \rangle \langle a_n \wedge a_n, \mathbf{e}_j \rangle. \quad (27)$$

Since

$$a - a_n = \varphi(\xi - \xi_n), \quad a_n \wedge a_n = \frac{1}{4} d\varphi \wedge d\varphi - d\varphi \wedge \xi_n,$$

and $\xi_n \rightarrow \xi$ in $W^{1,2}$, and φ has finite Faddeev energy, and both φ and u are bounded, the term

$$u^*(X_p(f^q)) \langle a - a_n, e_p \rangle \langle a_n \wedge a_n, e_q \rangle = u^*(X_p(f^q)) \varphi^p(\xi - \xi_n) \wedge \left(\frac{1}{4} \langle d\varphi \wedge d\varphi, e_q \rangle - d\varphi^q \wedge \xi_n \right)$$

converges to 0 in $L^{3/2}$. Notice that this would not work if a_n was simply a mollification of a because a_n would only approach a in L^2 and $a_n \wedge a_n$ only approach $a \wedge a$ in L^1 . With our approximation, the worst terms cancel.

Moving to the second term in (27) notice that for any 1-form b with values in \mathbb{R}^3 we can write

$$\begin{aligned} \langle b, \mathbf{e}_p \rangle &= b^p \\ b \wedge b &= 2 b^1 \wedge b^2 \mathbf{e}_3 + \text{cyclic} \\ \langle b, b \wedge b \rangle &= 6 b^1 \wedge b^2 \wedge b^3 \\ \langle b, \mathbf{e}_p \rangle \langle b \wedge b, \mathbf{e}_q \rangle &= \frac{1}{3} \langle b, b \wedge b \rangle \delta_{pq}. \end{aligned}$$

Using this, the second term in (27) becomes

$$\frac{1}{3} u^*(X_i(f^i)) \langle a_n, a_n \wedge a_n \rangle.$$

More cancelation takes place when we analyze this term. Indeed,

$$\langle a_n, a_n \wedge a_n \rangle = \left\langle -\frac{1}{2} \varphi d\varphi + \varphi \xi_n, \frac{1}{4} d\varphi \wedge d\varphi - d\varphi \wedge \xi_n \right\rangle = -\frac{3}{4} \xi_n \wedge \varphi d\varphi \wedge d\varphi.$$

Now, $\langle a, a \wedge a \rangle = -\frac{3}{4} \xi \wedge \varphi d\varphi \wedge d\varphi$. Since $\varphi d\varphi \wedge d\varphi \in L^2$ and $\xi_n \rightarrow \xi$ in L^6 , we conclude that

$$\langle a_n, a_n \wedge a_n \rangle \rightarrow \langle a, a \wedge a \rangle \quad \text{in } L^{3/2}.$$

Putting all parts together, with a slight abuse of notation we can write

$$\begin{aligned} \lim_{n \rightarrow \infty} d \langle u^* f, a_n \wedge a_n \rangle &= \lim_{n \rightarrow \infty} \langle d(u^* f), a_n \wedge a_n \rangle = \\ \lim_{n \rightarrow \infty} \frac{1}{3} u^*(X_i(f^i)) \langle a_n, a_n \wedge a_n \rangle &= \frac{1}{3} u^*(X_i(f^i)) \langle a, a \wedge a \rangle. \end{aligned}$$

This agrees with our simplification (26) of the right-hand side of (24) and completes the argument. Notice that $\langle a, a \wedge a \rangle = -\frac{3}{4} \xi \wedge \varphi d\varphi \wedge d\varphi$ gives a second proof that $\langle a, a \wedge a \rangle$ is in L^1 (in fact $L^{3/2}$).

□

Remark 20 *The decomposition $\langle a_n, a_n \wedge a_n \rangle = -\frac{3}{4} \xi_n \wedge \varphi d\varphi \wedge d\varphi$ would hold for any symmetric space, but it does not hold for homogeneous spaces [33]. Thus we expect that our argument would generalize to symmetric spaces, but an alternate argument would be required for homogeneous spaces.*

Example 8. A $W^{1,2}$ lift of a finite Faddeev energy map need not have finite Skyrme energy. Indeed, let φ be a $W^{1,2}$ map into a great circle on S^2 . Then $d\varphi \wedge d\varphi = 0$ and φ automatically has finite Faddeev energy. Take a real, closed 1-form $\xi \in L^2$ and set $a = -\frac{1}{2} \varphi d\varphi + \varphi \xi$. Clearly, $a \in L^2$. Because $d\xi = 0 = -\frac{1}{4} \varphi d\varphi \wedge d\varphi$, a is flat. Thus, there exists a $W^{1,2}$ lift u so that $u^{-1}du = a$. As we have seen in the proof of Proposition 18, $a \wedge a = \frac{1}{4} d\varphi \wedge d\varphi - d\varphi \wedge \xi$, which in the present case gives $u^{-1}du \wedge u^{-1}du = a \wedge a = -d\varphi \wedge \xi$. Now it is easy to find φ and ξ so that $d\varphi \wedge \xi$ is not in L^2 , and this would imply that u has infinite Skyrme energy. At the same time, $(u^{-1}du)^{\wedge 3} = 0 \in L^1$.

For Ω a neighborhood of the origin in \mathbb{R}^3 , here is a concrete example of φ and ξ that illustrates the above argument:

$$\varphi(x^1, x^2, x^3) = \mathbf{j} \exp(\mathbf{i}(x^3)^{3/5}), \quad \xi = r^{-6/5} dr.$$

By inserting smooth cut-off functions into the exponentials this example can be patched into \mathbb{R}^3 . Of course in this example we could have taken $\xi = 0$ and obtained a lift with finite Skyrme energy. It is an interesting open question whether every finite Faddeev energy map defined on a cube has a finite Skyrme energy lift.

3.3 Representation of S^2 -valued maps on \mathbb{R}^3

We now establish a version of our local representation proposition for maps $\varphi : \mathbb{R}^3 \rightarrow S^2$. We use it later to give a correct proof that the Hopf invariant of a finite Faddeev energy map from \mathbb{R}^3 to S^2 is an integer.

The essential difference between this and the previous case is that the domain is no longer compact, so the proof of Proposition 18 does not automatically carry over to this case. The step where we use the Hodge decomposition is slightly modified. The Hodge decomposition theorem is usually stated for compact manifolds, but it also holds when there is a suitable balance between the function spaces and the geometry of the manifold at infinity. The result that we need is presented below.

Lemma 21 *Any L^2 form on \mathbb{R}^3 can be expressed in a unique way as*

$$\alpha = d\xi + \delta\omega$$

with $\delta\xi = 0$, $d\omega = 0$, where ξ and ω are in L^6 , $d\xi$ and $\delta\omega$ are in L^2 .

This result is well known to people working in mathematical hydrodynamics.

We now state and prove the representation result.

Proposition 22 *If φ is a finite Faddeev energy map from \mathbb{R}^3 to S^2 then there is a map $u : \mathbb{R}^3 \rightarrow S^3$ with $du \in L^2$ so that $\varphi = u^{-1} \mathbf{i} u$. In addition, $u^{-1} du \wedge u^{-1} du \wedge u^{-1} du$ is in L^1 and $L^{3/2}$, and u has the following important property: For any smooth \mathbb{R}^3 -valued function f on S^3 ,*

$$u^* (d\langle f, y^{-1} dy \wedge y^{-1} dy \rangle) = d (u^* (\langle f, y^{-1} dy \wedge y^{-1} dy \rangle)) . \quad (28)$$

Here $y^{-1} dy$ is the usual Maurer-Cartan form on S^3 . One also has

$$u^* (d\langle f, y^{-1} dy \wedge y^{-1} dy \rangle) \in L^1(\mathbb{R}^3)$$

and

$$\int_{\mathbb{R}^3} u^* (d\langle f, y^{-1} dy \wedge y^{-1} dy \rangle) = 0 . \quad (29)$$

Furthermore, for any two such maps u and v there is a map $\lambda : \mathbb{R}^3 \rightarrow S^1$ with $d\lambda \in L^2(\mathbb{R}^3, S^1)$ so that $v = \lambda u$. One can choose u as above with

$$\delta \langle u^{-1} du, \varphi \rangle = 0, \quad \text{and} \quad d \langle u^{-1} du, \varphi \rangle \in L^2 \quad (30)$$

where δ is the codifferential. Such a lift is unique up to left multiplication by a unit complex number.

Proof. The proof is of course similar to the proof of Proposition 18. We therefore point out the modifications that are necessary to the previous proof. The first change comes about when looking for the 1-form ξ . Recall that ξ is defined as a solution of the elliptic system

$$d\xi = -\frac{1}{4} \varphi d\varphi \wedge d\varphi, \quad \delta \xi = 0. \quad (31)$$

The finite energy condition tells us that $-\frac{1}{4} \varphi d\varphi \wedge d\varphi$ is in L^2 . We also know that this form is closed (in the sense of distributions). It follows from the Hodge decomposition, Lemma 21, that system (31) has a unique solution $\xi = \xi_i dx^i$ with the following properties: first derivatives of each coefficient $\xi_i(x)$ are square-integrable, and each ξ_i is in L^6 . This means that $a = -\frac{1}{2} \varphi d\varphi + \varphi \xi$ will no longer be in L^2 , it will be in $L^2 + L^6$. However, a is in L^2 on any cube so we can find a sequence of lifts u_n defined on $[-n, n]^3$. The uniqueness assertion from the compact version of the theorem tells us that u_n and u_1 agree on the unit cube up to a factor in S^1 . Multiplying by this factor if necessary the uniqueness result tells us that the u_n agree on their domains. It follows that we can define a lift u on all of \mathbb{R}^3 .

We now just need to establish equations (28), (29) and that $u^{-1} du \wedge u^{-1} du \wedge u^{-1} du \in L^1$. The argument leading to

$$u^* (d\langle f, y^{-1} dy \wedge y^{-1} dy \rangle) = \frac{1}{3} u^* (X_m(f^m)) \langle a, a \wedge a \rangle, \quad (32)$$

works without modification. Turning to the left hand side of equation (28) we can only approximate ξ by a sequence ξ_n of smooth compactly supported forms such that $d\xi_n \rightarrow d\xi$ in L^2 and $\xi_n \rightarrow \xi$ in L^6 . We set

$$a_n := -\frac{1}{2} \varphi d\varphi + \varphi \xi_n,$$

obtaining $a_n \wedge a_n = \frac{1}{4} d\varphi \wedge d\varphi - d\varphi \wedge \xi_n$ and notice that

$$d\langle u^* f, a_n \wedge a_n \rangle \rightarrow d\langle u^* f, a \wedge a \rangle \quad \text{as distributions.}$$

Also because it is a local statement (i.e., we can fix a test function and then work in a compact set containing the support of that test function and apply the argument from the compact case),

$$d\langle u^* f, a_n \wedge a_n \rangle \rightarrow u^* (d\langle f, y^{-1} dy \wedge y^{-1} dy \rangle) \quad \text{as distributions.}$$

To establish that $u^{-1} du \wedge u^{-1} du \wedge u^{-1} du \in L^{3/2}$ we cannot restrict to a compact set, but the decomposition $\langle a, a \wedge a \rangle = -\frac{3}{4} \xi \wedge \varphi d\varphi \wedge d\varphi$ still holds and this is good enough as $\xi \in L^6$, φ is bounded and $d\varphi \wedge d\varphi \in L^2$. To see that $u^{-1} du \wedge u^{-1} du \wedge u^{-1} du \in L^1$, notice that $d\varphi \in L^2$ implies that $d\varphi \wedge d\varphi \in L^1$ so $d\varphi \wedge d\varphi \in L^{6/5}$ by interpolation. Combined with the decomposition of $\langle a, a \wedge a \rangle$, this gives the result.

Notice that together with equation (32) this implies that $u^* (d\langle f, y^{-1} dy \wedge y^{-1} dy \rangle)$ is in L^1 . Now $a \wedge a = \frac{1}{4} d\varphi \wedge d\varphi - d\varphi \wedge \xi$ is in $L^{3/2}$ as $d\varphi \in L^2$, $\xi \in L^6$ and the first term is in both L^1 and L^2 . Working in spherical coordinates (r, ϑ) this implies that there is a sequence $R_n \rightarrow \infty$ such that

$$\int_{S^2} |\langle u^* f, a \wedge a \rangle(R_n, \vartheta)|^{3/2} d\text{vol}_{S^2} \leq 1/(R_n^3 \ln R_n).$$

We now have

$$\begin{aligned} \left| \int_{\mathbb{R}^3} u^* (d\langle f, y^{-1} dy \wedge y^{-1} dy \rangle) \right| &= \left| \lim_{n \rightarrow \infty} \int_{D_{R_n}} d\langle u^* f, a \wedge a \rangle \right| \\ &= \left| \lim_{n \rightarrow \infty} \int_{\partial D_{R_n}} \langle u^* f, a \wedge a \rangle \right| \\ &\leq \lim_{n \rightarrow \infty} R_n^2 \int_{S^2} |\langle u^* f, a \wedge a \rangle(R_n, \vartheta)| d\text{vol}_{S^2} \\ &\leq \lim_{n \rightarrow \infty} R_n^2 \left(\int_{S^2} |\langle u^* f, a \wedge a \rangle(R_n, \vartheta)|^{3/2} d\text{vol}_{S^2} \right)^{2/3} (4\pi)^{1/3} = 0. \end{aligned}$$

□

4 The primary invariant

In this section, we define the pull-back map on second cohomology for finite Faddeev energy maps. We also prove that two S^2 -valued maps with at least this much regularity and the same induced map on the second cohomology H^2 are related by a family of isometries of S^2 . This is a generalization of Theorem 10 ([6, Lemma 1]) which proves the same result for smooth maps.

4.1 Definition

The first problem involved in generalizing Pontrjagin's theorem for Sobolev maps is to identify a suitable cohomology theory where one can associate an element to each S^2 -valued map. While the third cohomology of any oriented 3-manifold is isomorphic to the integers, the second cohomology of an oriented 3-manifold may have torsion, e.g. $H^2(\mathbb{R}P^3; \mathbb{Z}) \cong \mathbb{Z}_2$. Thus the deRham model is no longer sufficient. We use Čech cohomology. Details about Čech theory can be found in [45]. Recall the natural way to define pull-backs in Čech theory. Given a continuous map $\varphi : X \rightarrow Y$ and acyclic open covers $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ and $\mathcal{V} = \{V_\beta\}_{\beta \in \mathcal{B}}$ of X and Y respectively such that for every $U_\alpha \in \mathcal{U}$, there is a $V_\beta \in \mathcal{V}$ such that $\varphi(U_\alpha) \subseteq V_\beta$, one can define a pull-back on the Čech cohomology. (Notice that such covers exist for any continuous map, since an acyclic cover on X may be refined by intersecting it with the inverse image of the acyclic cover on Y . It is not obvious how to do the analogous construction with a Sobolev map.) If μ is the Čech k -cycle on Y represented by the collection of integers $m_{\beta_0 \dots \beta_k}$, the pull-back $\varphi^* \mu$ is defined to be the class represented by the collection of integers $n_{\alpha_0 \dots \alpha_k} = \sum_{\varphi(U_{\alpha_j}) \subseteq V_{\beta_j}} m_{\beta_0 \dots \beta_k}$.

As we now show, we can fix a cover on M and define a class $\varphi^* \mu_{S^2} \in \check{H}^2(M; \mathbb{Z})$ for every finite Faddeev energy map φ (without changing the cover) such that it will agree with the usual notion for smooth maps φ . The idea here is to construct a cover by cubes with cubic intersections and apply the local lifting proposition from the previous section to the map restricted to each cube. The lifts over the various cubes cannot agree on all of the overlaps (unless $\varphi^* \mu_{S^2} = 0$) so there must be circle-valued maps relating the lifts. These circle-valued lifts form a Čech cocycle representing a class in $\check{H}^1(M, S^1)$. Finally, we use the isomorphism of the first cohomology with coefficients in the multiplicative group S^1 with the second cohomology with integer coefficients, $H^1(M; S^1) \cong H^2(M; \mathbb{Z})$, to obtain our definition of the class $\varphi^* \mu_{S^2}$.

For the constructions below we need a triangulation of our 3-manifold M such that any nonempty intersection of closed stars of vertices is piecewise smoothly equivalent to the unit cube in \mathbb{R}^3 . We used such a triangulation in [5] as well. See Remark 14 for further discussion.

Definition 23 *A triangulation of a manifold is called cube-like if any nonempty intersection of closed stars of vertices is piecewise smoothly equivalent to the unit cube in \mathbb{R}^n .*

Fix a cube-like triangulation. Let U_p denote the open star of the vertex p , U_{pq} denote the intersection $U_p \cap U_q$, $U_{pqr} = U_p \cap U_q \cap U_r$, etc.

Given a finite energy map $\varphi : M \rightarrow S^2$, for every U_p pick a local lift $u_p : U_p \rightarrow S^3$ so that $\varphi = u_p^{-1} \mathbf{i} u_p$ and $u_p \in W^{1,2}(U_p; S^3)$. That such a lift exists is guaranteed by Proposition 18. Set $\lambda_{pq} = u_p u_q^{-1}$ on U_{pq} . By Proposition 18, this is an S^1 -valued map and $\lambda_{pq} \in W^{1,2}(U_{pq}; S^1)$. Clearly, $\lambda_{qp} = \bar{\lambda}_{pq} = \lambda_{pq}^{-1}$. On the non-empty triple intersections U_{pqr} the functions λ_{pq} satisfy the cocycle condition

$$\lambda_{qr} \lambda_{rp} \lambda_{pq} = 1. \quad (33)$$

The maps λ_{pq} can be written in the form

$$\lambda_{pq}(x) = e^{2\pi \mathbf{i} \theta_{pq}(x)},$$

for some real-valued functions $\theta_{pq} \in W^{1,2}(U_{pq}; \mathbb{R})$. This follows from [12, Lemma 1], or from our Lemma 12 (after noticing that the $\mathbf{i}\mathbb{R}$ -valued 1-form $\lambda_{pq}^{-1} d\lambda_{pq}$ is in L^2). Define the function

$$n_{pqr} = \theta_{qr} + \theta_{rp} + \theta_{pq} \quad (34)$$

on U_{pqr} . In view of (33), $\exp(2\pi \mathbf{i} n_{pqr}) = 1$, so n_{pqr} must take integer values. If the map φ were smooth, it would follow that the functions n_{pqr} were smooth and hence constant (as is any mapping from a connected set into a discrete space). With Sobolev maps the situation is more subtle since such maps can be discontinuous. However, our functions are in $W^{1,1}$ and this is enough. Indeed the following result allows one to conclude that the image of a subset of full measure in U_{pqr} is connected.

Proposition 24 *If X is a smooth, compact, connected manifold, Y is a compact subset of \mathbb{R}^N , and $u \in W^{1,1}(X, Y)$, then there is a connected component Y_u of Y so that $u(x) \in Y_u$ for almost every x .*

This result is proved in Appendix 6. Alternatively, we could have quoted a result of Giaquinta-Modica-Souček [23] that is sufficient. Thus, there is a fixed integer that we will also denote n_{pqr} such that $n_{pqr}(x) = n_{pqr}$ for almost all $x \in \overline{U}_{pqr}$. It is not hard to see that

$$n_{pqr} - n_{\ell qr} + n_{\ell pr} - n_{\ell pq} = 0, \quad (35)$$

provided $U_{\ell pqr} \neq \emptyset$. Thus, n_{pqr} defines a Čech 2-cocycle in the constant sheaf \mathbb{Z} .

Definition 25 *Given $\varphi \in W^{1,2}(M, S^2)$, we call u_p , λ_{pq} , θ_{pq} , and n_{pqr} its local representatives, transition functions, lifted transition functions, and cocycle, respectively. The primary homotopy invariant is the cohomology class represented by n_{pqr} and it is denoted by $\varphi^* \mu_{S^2}$.*

Remark 26 *Notice that this defines the pull-back for fairly singular functions such as those from Example 5. This approach to define $\varphi^* \mu_{S^2}$ would not work for $\varphi \in W^{1,p}(M, S^2)$ for $p < 3$ without additional regularity assumptions. Indeed, it is unlikely that there is any reasonable definition of the pull-back for such functions in light of the function $\phi : D^3 \rightarrow S^2$ from Example 4 given by $\phi(x) = x/|x|$.*

Since the local representatives are not unique, we should explain why $\varphi^* \mu_{S^2}$ is well-defined. By Proposition 18, any other set of representatives is obtained by multiplying each u_p on the left by some S^1 -valued function $\mu_p \in W^{1,2}$. We thus write another set of representatives, $\tilde{u}_p = \mu_p u_p$, and another set of transition functions, $\tilde{\lambda}_{pq} = \tilde{u}_p \tilde{u}_q^{-1}$. The transition functions $\tilde{\lambda}_{pq}$ are related to the transition functions λ_{pq} by the equation

$$\lambda_{pq} \tilde{\lambda}_{pq}^{-1} = \mu_p^{-1} \mu_q,$$

meaning, of course, that $\lambda_{pq} \tilde{\lambda}_{pq}^{-1}$ is a coboundary. The choice of the lifts θ_{pq} also fails to be unique: we can replace $\theta_{pq}(x)$ by $\widehat{\theta}_{pq}(x) = \theta_{pq}(x) + \gamma_{pq}$ with integer γ_{pq} . Defining ξ_p by $\mu_p^{-1} d\mu_p = 2\pi i d\xi_p$, and incorporating γ_{pq} , we obtain the equation relating the lifted transition functions: $\tilde{\theta}_{pq}(x) = \theta_{pq}(x) + \gamma_{pq} + \xi_p(x) - \xi_q(x)$. The effect this makes on the cocycle n_{pqr} is this:

$$\tilde{n}_{pqr} = n_{pqr} + \gamma_{pq} + \gamma_{qr} + \gamma_{rp}.$$

This equation represents the fact that \tilde{n}_{pqr} and n_{pqr} belong to the same 2-cohomology class.

This extension of the pull-back of the fundamental class to Sobolev maps satisfies everything that one could hope for. The next proposition verifies that it coincides with the usual definition for smooth maps and that every cohomology class is represented as the primary invariant of some smooth map. The proposition afterward extends an important lifting result from the smooth case.

Proposition 27 *The class $\varphi^* \mu_{S^2}$ given in Definition 25 exactly coincides with the usual definition for smooth maps φ . Furthermore, given any Sobolev map for which $\varphi^* \mu_{S^2}$ is defined, there is a smooth map with the same value.*

Proof. We first identify the associated cohomology class in the smooth case. We have $\varphi = u_p^{-1} i u_p$. In the smooth case the u_p 's are sections of the bundle $Q_{\varphi i} = \{(x, q) \in M \times S^3 \mid i = q \varphi(x) q^{-1}\}$. It follows that $\lambda_{pq} = u_p u_q^{-1}$ are the transition functions of the complex line bundle associated with the principal bundle $Q_{\varphi i}$. The cocycle n_{pqr} represents (minus) the first Chern class of the line bundle associated to $Q_{\varphi i}$, [26]. By this and the proof of [6, Lemma 1], we have

$$[n_{pqr}] = c_1(Q_{\varphi i}) = \varphi^* \mu_{S^2} - i^* \mu_{S^2} = \varphi^* \mu_{S^2}.$$

It turns out that every second cohomology class $\beta \in H^2(M; \mathbb{Z})$ is represented as $\varphi^* \mu_{S^2}$ for some smooth map $\varphi : M \rightarrow S^2$. We give two arguments for this, one that quotes well-known results from topology and one that follows from the definition. The first argument uses that $H^2(M; \mathbb{Z})$ is in one-to-one correspondence with the homotopy classes of maps from M to the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$ which is $\mathbb{C}P^\infty$. Now the homotopy classes of maps into $\mathbb{C}P^\infty$ is the same as the homotopy classes of maps into $\mathbb{C}P^2$ by general position. It follows that any given class in $H^2(M; \mathbb{Z})$, may be represented as the pull-back of a map $f : M \rightarrow \mathbb{C}P^2$. By general position this map may be assumed to miss $[0 : 0 : 1]$. Composing with the natural projection $\mathbb{C}P^2 - \{[0 : 0 : 1]\} \rightarrow S^2$ gives the desired map.

The second argument is more direct. Let $\alpha \in H^2(M; \mathbb{Z})$ be given. All we need is to construct a continuous map $\tilde{\varphi}$ so that $\tilde{\varphi}^* \mu_{S^2} = \alpha$, because this $\tilde{\varphi}$ can be continuously deformed into a smooth map φ , and the cohomology class will not change under homotopy. The continuous map is defined on the triangulated M as follows. First, map the whole 1-skeleton into a fixed point $p \in S^2$. Next, define a map on 2-simplices. Pick a functional on 2-chains n representing α . For each 2-simplex σ^2 , take any degree $n(\sigma^2)$ map from $\sigma^2 / \partial \sigma^2$ into S^2 with the boundary, $\partial \sigma^2$, going into p . This defines a map on the 2-skeleton. The resulting map can be extended to 3-simplices σ^3 because n is closed, i.e., $n(\partial \sigma^3) = 0$ for any σ^3 . (This is just the 2-cocycle condition (35).) Thus, we obtain $\tilde{\varphi}$. \square

4.2 Global intertwining maps

We can now prove a global lifting result using the primary homotopy invariant. It is a generalization of [6, Lemma 1] to Sobolev maps.

Theorem 28 *For two finite energy maps $\varphi, \psi : M \rightarrow S^2$ to be intertwined,*

$$\varphi(x) = \Phi(x) \psi(x) \Phi(x)^{-1}, \quad (36)$$

by a map $\Phi \in W^{1,2}(M, S^3)$ it is necessary and sufficient that $\varphi^ \mu_{S^2} = \psi^* \mu_{S^2}$.*

If $\varphi^ \mu_{S^2} = \psi^* \mu_{S^2}$ **and in addition ψ is smooth**, then there exists a **cartesian** intertwining map $\Phi \in W^{1,2}(M, S^3)$, i.e., in addition to (36), $\text{Re}(\Phi^{-1} d\Phi \wedge \Phi^{-1} d\Phi \wedge \Phi^{-1} d\Phi) \in L^1(M)$ (it is in fact in $L^{3/2}$) and, for any smooth \mathbb{R}^3 -valued function f on S^3 ,*

$$\Phi^* (d\langle f, y^{-1} dy \wedge y^{-1} dy \rangle) = d(\Phi^* (\langle f, y^{-1} dy \wedge y^{-1} dy \rangle)) , \quad (37)$$

on M in the sense of distributions. Furthermore $(\Phi^{-1} d\Phi)^{\wedge 2}$ is in $L^{3/2}$ and $\delta\langle \Phi^{-1} d\Phi, \psi \rangle = 0$.

If Φ_1 and Φ_2 are two different intertwining maps, then there is a λ in $W^{1,2}(M, S^1)$ such that $\Phi_1(x) \mathbf{q}(\psi(x), \lambda(x)) = \Phi_2(x)$, where \mathbf{q} is the map from $S^2 \times S^1$ to S^3 defined by the formula

$$\mathbf{q}(x, \lambda) = q^{-1} \lambda q, \quad \text{for any } q \in S^3 \text{ such that } x = q^{-1} \mathbf{i} q. \quad (38)$$

If $\delta\langle \Phi_k^{-1} d\Phi_k, \psi \rangle = 0$ for $k = 1, 2$ then $\langle \lambda^{-1} d\lambda, \mathbf{i} \rangle$ is harmonic and λ is smooth.

Proof. We work with a cube-like triangulation K . First assume that there is a map Φ with $\varphi = \Phi \psi \Phi^{-1}$. Let v_p be the local representatives for ψ . It follows that $u_p = v_p \Phi^{-1}$ are local representatives for φ . The corresponding transition functions $\lambda_{pq} = v_p \Phi^{-1} \Phi v_q^{-1}$ agree identically with the transition functions for ψ . It follows that $\psi^* \mu_{S^2} = \varphi^* \mu_{S^2}$.

Now assume that $\psi^* \mu_{S^2} = \varphi^* \mu_{S^2}$ and construct an intertwining map Φ . We patch it together from local pieces. Let $u_p, \lambda_{pq}, \theta_{pq}, n_{pqr}$ ($v_p, \kappa_{pq}, \vartheta_{pq}, m_{pqr}$) be the the local representatives, transition functions, lifted transition functions and cocycle corresponding to φ (ψ respectively). Recall that this means that

$$\begin{aligned} \varphi &= u_p^{-1} \mathbf{i} u_p, \\ \lambda_{pq} &= u_p u_q^{-1} = \exp(2\pi \mathbf{i} \theta_{pq}), \\ n_{pqr} &= \theta_{qr} + \theta_{rp} + \theta_{pq} \quad \text{a.e.}, \end{aligned}$$

and similarly for ψ . Also recall from Proposition 18 that we can take $u_p \in W^{1,2}$ satisfying the gauge-fixing condition

$$\delta\langle u_p^{-1} du_p, \varphi \rangle = 0, \quad \text{and} \quad i^* * (\langle u_p^{-1} du_p, \varphi \rangle) = 0.$$

In addition, the real 1-form $\xi_p = \langle u_p^{-1} du_p, \varphi \rangle$ is the solution of the problem

$$d\xi_p = -\frac{1}{4} \varphi d\varphi \wedge d\varphi, \quad \delta\xi_p = 0,$$

with the boundary condition $i^* * \xi_p = 0$. Notice that on the intersection U_{pq} the difference $\xi_p - \xi_q$ is a harmonic form,

$$d(\xi_p - \xi_q) = 0, \quad \delta(\xi_p - \xi_q) = 0,$$

and hence, $\xi_p - \xi_q$ is smooth on U_{pq} . The lifted transition functions θ_{pq} satisfy the equation

$$2\pi d\theta_{pq} = \langle \mathbf{i}, \lambda_{pq}^{-1} d\lambda_{pq} \rangle = \langle \varphi, u_p^{-1} du_p \rangle - \langle \varphi, u_q^{-1} du_q \rangle = \xi_p - \xi_q.$$

Since U_{pq} is piecewise smoothly equivalent to a cube, a smooth solution, θ_{pq} , can be written down “explicitly”. In the appropriate coordinate system (i.e., in the cube, with x_0 inside the cube)

$$\theta_{pq}(x) = \theta_{pq}(x_0) + \frac{1}{2\pi} \int_0^1 (\xi_p - \xi_q)_j(tx) x^j dt. \quad (39)$$

Thus, the functions θ_{pq} are C^∞ . Any other solution of the equation

$$2\pi d\theta_{pq} = \xi_p - \xi_q \quad (40)$$

differs from (39) by a constant. It should be noted that we cannot claim that θ_{pq} is C^∞ on the closure of U_{pq} . The best we can say about the behavior of θ_{pq} up to the boundary is that $\theta_{pq} \in W^{2,2}(U_{pq})$. Indeed, since both ξ_p and ξ_q are in $W^{1,2}(U_{pq})$, we get θ_{pq} in $W^{1,2}(U_{pq})$ from (39). Differentiating equation (40) shows that the second derivatives of θ_{pq} are in $L^2(U_{pq})$.

In a similar fashion, we have initially $\lambda_{pq} \in W^{1,2}(U_{pq})$. However, differentiating the equation $\lambda_{pq} = \exp(2\pi \mathbf{i} \theta_{pq})$ twice (and using the fact that the $L^4(U_{pq})$ -norm of $d\theta_{pq}$ is bounded due to Sobolev imbedding $W^{1,2} \subset L^4$ in the three dimensional case), we conclude that $\lambda_{pq} \in W^{2,2}(U_{pq})$. In the interior of U_{pq} the function λ_{pq} is C^∞ , of course.

Let us mention here that, because U_{pq} is bilipschitz equivalent to a cube, its boundary satisfies the minimal regularity condition that allows us to extend $W^{2,2}$ functions from U_{pq} to the whole manifold. In fact, there exists a bounded extension operator $E_{pq} : W^{2,2}(U_{pq}) \rightarrow W^{2,2}(M)$, see [46, Chapter 6] and [1]. Also, in dimension three the $W^{2,2}$ functions are continuous (actually, Hölder continuous). Thus, in particular, the functions $n_{pqr}(x) = \theta_{pq}(x) + \theta_{qr}(x) + \theta_{rp}(x)$ are constant everywhere on U_{pqr} .

The same comments hold and we make similar choices for v_p , κ_{pq} , ϑ_{pq} , m_{pqr} .

The definition of the local representatives imply that to have $\varphi = \Phi \psi \Phi^{-1}$ locally, we must define $\Phi_p = u_p^{-1} \mu_p v_p$, for some $\mu_p : U_p = \text{st}(p) \rightarrow S^1$. In order for Φ_p and Φ_q to agree on U_{pq} , we must have

$$\mu_p \mu_q^{-1} = \lambda_{pq} \kappa_{pq}^{-1}. \quad (41)$$

In other words, the maps Φ_p agree on the overlaps and define a global map on M exactly when the 1-cochain $\lambda_{pq} \kappa_{pq}^{-1}$ is a coboundary. That this chain is a coboundary in turn will follow from our assumption $\psi^* \mu_{S^2} = \varphi^* \mu_{S^2}$. The desired maps μ_p are given in equation (44) below.

By quoting results from algebraic topology, we can arrive at a fast proof that this chain is a coboundary. However we need a bit more in order to obtain all of the statements of the

theorem. We thus sketch the argument from algebraic topology and then present the full proof by unwinding the quoted results in our particular case.

The short exact sequence of presheaves,

$$0 \rightarrow C^0(-, \mathbb{Z}) \rightarrow W^{1,2}(-, \mathbb{R}) \rightarrow W^{1,2}(-, S^1) \rightarrow 0,$$

gives a short exact sequence of chain complexes

$$0 \rightarrow \check{C}^*(M; \mathbb{Z}) \rightarrow \check{C}^*(M; W^{1,2}(-, \mathbb{R})) \rightarrow \check{C}^*(M; W^{1,2}(-, S^1)) \rightarrow 0,$$

that leads to a long exact sequence in cohomology. Since the class $\lambda\kappa^{-1}$ maps to $[n-m] = 0$ we see that it must be in the image of $\check{H}^*(M; W^{1,2}(-, \mathbb{R}))$. Using a partition of unity we see that $W^{1,2}(-, \mathbb{R})$ is a fine presheaf so that $\check{H}^*(M; W^{1,2}(-, \mathbb{R})) = 0$. It follows that $\lambda\kappa^{-1}$ is a coboundary. We now spell this out in greater detail.

The equality of the two classes $\psi^*\mu_{S^2}$, $\varphi^*\mu_{S^2}$ means that m_{pqr} differs from n_{pqr} only by $\gamma_{qr} + \gamma_{rp} + \gamma_{pq}$, where γ_{ij} are integers. Since

$$n_{pqr} = \theta_{qr} + \theta_{rp} + \theta_{pq},$$

by replacing θ_{pq} by $\theta_{pq} + \gamma_{pq}$ we can assume that $m_{pqr} = n_{pqr}$. This is allowed because θ_{pq} were only defined up to an integer choice ($\lambda_{pq}(x) = e^{2\pi i \theta_{pq}(x)}$) and the class $\varphi^*\mu_{S^2}$ is independent of all choices. This implies that

$$(\theta_{qr} - \vartheta_{qr}) + (\theta_{rp} - \vartheta_{rp}) + (\theta_{pq} - \vartheta_{pq}) = 0.$$

In other words, the family of real-valued functions $\beta_{pq}(x) = \theta_{pq}(x) - \vartheta_{pq}(x)$ forms a 1-cocycle: on U_{pqr} ,

$$\beta_{qr}(x) + \beta_{rp}(x) + \beta_{pq}(x) = 0. \quad (42)$$

Up to here we have just reproduced the part of the argument stating that since the class $\lambda\kappa^{-1}$ maps to $[n-m] = 0$ it must be in the image of $\check{H}^*(M; W^{1,2}(-, \mathbb{R}))$. We even know a bit more: the functions β_{pq} are smooth by our previous analysis of regularity.

We now turn to the part of the argument that uses the fine condition. This is an abstract way to say that chains are sums of chains with small support. Let ρ_k be a partition of unity subordinate to the cover $\{U_k\}$ and notice that $\beta_{pq} = \sum_k \rho_k \beta_{pq}$. It is clear that each $\rho_k \beta$ is coclosed. We will prove that they are all coexact as well.

We first recall that each β_{pq} can be extended to a function $\tilde{\beta}_{pq} = E_{pq}(\beta_{pq})$ on the whole manifold M so that

$$\|\tilde{\beta}_{pq}\|_{W^{2,2}(M)} \leq C \|\beta_{pq}\|_{W^{2,2}(U_{pq})}.$$

Define functions ς_p^k on U_p of class $W^{2,2}(U_p)$ by

$$\varsigma_p^k(x) := \begin{cases} \rho_k(x) \tilde{\beta}_{pk}(x) & \text{if } x \in U_k \\ 0 & \text{otherwise} \end{cases}$$

To see that

$$\rho_k \beta_{pq}(x) = \varsigma_p^k(x) - \varsigma_q^k(x) \quad \text{on } U_{pq}, \quad (43)$$

notice that both sides are zero if x is not in U_k . If $x \in U_k$ then $x \in U_{pqk}$ so the coclosed condition (42) gives

$$\rho_k \beta_{pq}(x) = \rho_k \beta_{pk}(x) + \rho_k \beta_{kq}(x) = \zeta_p^k(x) - \zeta_q^k(x).$$

Set

$$\varsigma_p(x) = \sum_k \zeta_p^k(x).$$

To complete the construction of the map Φ , it remains to set

$$\mu_p = \exp 2\pi \mathbf{i} \varsigma_p \quad (44)$$

and notice that equation (41) follows from equation (43) by exponentiation. The fact that $\Phi \in W^{1,2}(M; S^3)$ follows from local considerations.

To prove the next statement of the theorem (the cartesian property), we start by observing that μ_p and $\mu_p^{-1} = \mu_p^*$ both belong to the Sobolev space $W^{2,2}(U_p)$. It will be convenient to slightly change our view of μ_p by writing $\Phi_p = \Phi|_{U_p} = (\mu_p^{-1} u_p)^{-1} v_p$. We now assume that ψ is smooth and show that the map Φ we have just constructed is cartesian. The proof is very similar to the proof in Proposition 18. It will be sufficient to show that each local map Φ_p is cartesian. Using $u_p^{-1} du_p = -\varphi d\varphi + \varphi \xi_p$ and $u_p^{-1} \mathbf{i} u_p = \varphi$, we obtain

$$(\mu_p^{-1} u_p)^{-1} d(\mu_p^{-1} u_p) = u_p^{-1} du_p - \varphi (2\pi d\varsigma_p) = \frac{1}{2} \varphi^{-1} d\varphi + \varphi (\xi_p - 2\pi d\varsigma_p). \quad (45)$$

Notice that the real-valued 1-form $\xi_p - 2\pi d\varsigma_p \in W^{1,2}(U_p)$.

We fix a p and simplify the notation writing:

$$w = \mu_p^{-1} u_p, \quad v = v_p, \quad \Phi = \Phi_p = w^{-1} v,$$

and,

$$\xi = \xi_p - 2\pi d\varsigma_p, \quad a = w^{-1} dw, \quad b = v^{-1} dv.$$

Since ψ is smooth, v and b are smooth. On the other hand a is in L^2 and $a \wedge a$ is in $L^{3/2}$. We compute:

$$\Phi^{-1} d\Phi = b - \Phi^{-1} a \Phi, \quad (46)$$

$$(\Phi^{-1} d\Phi)^{\wedge 3} = b^{\wedge 3} - a^{\wedge 3} + 3 \operatorname{Re} (b \wedge b \wedge \Phi^{-1} a \Phi - b \wedge \Phi^{-1} a \wedge a \Phi). \quad (47)$$

The same argument used in the proof of Proposition 18 (namely decomposing a into components parallel and perpendicular to φ) shows that $a \wedge a \wedge a$ belongs to $L^{3/2}(U_p)$. The form $b \wedge b \wedge b$ belongs to $L^{3/2}(U_p)$ because it is the restriction of a smooth function defined on $\overline{U_p}$. Since a is in L^2 , $a \wedge a$ is in $L^{3/2}$, Φ is bounded and the rest of the factors in the remaining term are smooth we see that $(\Phi^{-1} d\Phi)^{\wedge 3}$ is in $L^{3/2}$ as claimed.

Given a smooth function f from S^3 to \mathbb{R}^3 we need to prove the equality

$$\Phi^* (d\langle f, y^{-1} dy \wedge y^{-1} dy \rangle) = d(\Phi^* (\langle f, y^{-1} dy \wedge y^{-1} dy \rangle)) ,$$

The proof is a modification of the argument used in Proposition 18 – we construct a sequence that approaches both sides of the equation in the sense of distributions. Recall that on S^3 ,

$$d\langle f, y^{-1} dy \wedge y^{-1} dy \rangle = \frac{1}{3} \sum_{i=1}^3 X_i(f^i) \langle y^{-1} dy, y^{-1} dy \wedge y^{-1} dy \rangle,$$

and hence the left-hand side is

$$\Phi^* d\langle f, y^{-1} dy \wedge y^{-1} dy \rangle = \frac{1}{3} \Phi^* \left(\sum_{i=1}^3 X_i(f^i) X_i(h^i) \right) \langle c, c \wedge c \rangle,$$

where

$$c := \Phi^{-1} d\Phi = b - \Phi^{-1} b \Phi.$$

Also,

$$\Phi^* (\langle f, y^{-1} dy \wedge y^{-1} dy \rangle) = \langle \Phi^*(f), c \wedge c \rangle.$$

To construct our sequence of approximations let ξ_n be a sequence of smooth 1-forms converging to ξ in $W^{1,2}$, and denote

$$a_n := \frac{1}{2} \varphi^{-1} d\varphi + \varphi \xi_n, \quad c_n := b - \Phi^{-1} a_n \Phi. \quad (48)$$

We have

$$c_n \wedge c_n := \frac{1}{4} \Phi^{-1} d\varphi \wedge d\varphi \Phi - \Phi^{-1} d\varphi \wedge \xi_n \Phi - b \wedge \Phi^{-1} a_n \Phi - \Phi^{-1} a_n \Phi \wedge b + b^{\wedge 2}. \quad (49)$$

Note that

$$c - c_n = \Phi^{-1} \varphi (\xi_n - \xi) \Phi,$$

converges to zero in $W^{1,2}$ and

$$c^{\wedge 2} - c_n^{\wedge 2} = \Phi^{-1} d\varphi \wedge (\xi_n - \xi) \Phi + \Phi^{-1} \varphi (\xi_n - \xi) \Phi \wedge b + b \wedge \Phi^{-1} \varphi (\xi_n - \xi) \Phi, \quad (50)$$

converges to zero in $L^{3/2}$. This establishes that $(\Phi^{-1} d\Phi)^{\wedge 2}$ is in $L^{3/2}$. This also implies that $\langle \Phi^* f, c_n \wedge c_n \rangle \rightarrow \langle \Phi^* f, c \wedge c \rangle$ in $L^{3/2}$, so clearly

$$d\langle \Phi^* f, c_n \wedge c_n \rangle \rightarrow d\langle \Phi^* f, c \wedge c \rangle \quad \text{as distributions.}$$

By the product rule

$$d\langle \Phi^* f, c_n \wedge c_n \rangle = \langle d(\Phi^* f), c_n \wedge c_n \rangle + \langle \Phi^* f, d(c_n \wedge c_n) \rangle \quad (51)$$

We need to justify this application of the product rule. As usual we approximate the various terms. The key to this step is just to notice that when computing

$$d(c_n \wedge c_n) = d(c_n \wedge c_n - c \wedge c),$$

using equation (50) the answer tends to zero in L^1 . To prove this convergence we use (50) and compute

$$\begin{aligned} d(c_n^{\wedge 2} - c^{\wedge 2}) &= d(\Phi^{-1}(d\varphi \wedge (\xi - \xi_n) + \varphi(\xi - \xi_n) \wedge \Phi b \Phi^{-1} + \Phi b \Phi^{-1} \varphi(\xi - \xi_n)) \Phi) \\ &= [c_n^{\wedge 2} - c^{\wedge 2}, \Phi^{-1} d\Phi] \\ &\quad + \Phi^{-1} d(d\varphi \wedge (\xi - \xi_n) + \varphi(\xi - \xi_n) \wedge \Phi b \Phi^{-1} + \Phi b \Phi^{-1} \varphi(\xi - \xi_n)) \Phi. \end{aligned} \quad (52)$$

We have $\Phi^{-1} d\Phi = b - \Phi^{-1} a \Phi$. Since b is smooth and bounded and $c_n^{\wedge 2} - c^{\wedge 2} \rightarrow 0$ in $L^{3/2}$ the term $[c_n^{\wedge 2} - c^{\wedge 2}, b] \rightarrow 0$ in $L^{3/2}$ and hence in L^1 . Using equation (50) and $a = \frac{1}{2} \varphi^{-1} d\varphi + \varphi \xi$ we also have

$$[c_n^{\wedge 2} - c^{\wedge 2}, \Phi^{-1} a \Phi] = \Phi^{-1} [d\varphi \wedge (\xi - \xi_n) + \varphi(\xi - \xi_n) \wedge \Phi b \Phi^{-1} + \Phi b \Phi^{-1} \varphi(\xi - \xi_n), \frac{1}{2} \varphi^{-1} d\varphi + \varphi \xi] \Phi.$$

Since φ , b and Φ are bounded and $\xi_n \rightarrow \xi$ in $W^{1,2}$ hence in L^6 , and $d\varphi$ is in L^2 the only troublesome looking term in this equation is $[d\varphi \wedge (\xi - \xi_n), \frac{1}{2} \varphi^{-1} d\varphi]$. However this last commutator can be arranged into a bounded factor times $d\varphi \wedge d\varphi \wedge (\xi - \xi_n)$ and $d\varphi \wedge d\varphi$ is in L^2 since φ has finite Faddeev energy. Thus $[c_n^{\wedge 2} - c^{\wedge 2}, \Phi^{-1} a \Phi] \rightarrow 0$ in $L^{6/5}$ and hence L^1 . For the last term in equation (52) notice that $dd\varphi = 0$, $d\varphi$ and $d\Phi$ are in L^2 , db is smooth and bounded, and $\xi_n \rightarrow \xi$ in $W^{1,2}$ to conclude that the entire last term converges to zero in L^1 . This completes the argument showing that $d(c_n \wedge c_n) \rightarrow 0$ in L^1 and thus justifying the use of the product rule in equation (51). It also shows that the second term of equation (51) tends to zero in L^1 . We now turn to the first term of (51).

Using the notation $\mathbf{e}_1 = \mathbf{i}$ etc. introduced in the proof of Proposition 18 and summation convention, we write

$$d(\Phi^* f) = \Phi^* (X_m(f^\ell)) \langle c, \mathbf{e}_m \rangle \mathbf{e}_\ell.$$

Thus,

$$\langle d(\Phi^* f), (c \wedge c)_n \rangle = \Phi^* (X_m(f^\ell)) \langle c, \mathbf{e}_m \rangle \wedge \langle (c \wedge c)_n, \mathbf{e}_\ell \rangle$$

We notice that

$$\langle c, \mathbf{e}_m \rangle \wedge \langle (c \wedge c)_n, \mathbf{e}_\ell \rangle = \langle c - c_n, \mathbf{e}_m \rangle \wedge \langle c_n \wedge c_n, \mathbf{e}_\ell \rangle + \langle c_n, \mathbf{e}_m \rangle \wedge \langle c_n \wedge c_n, \mathbf{e}_\ell \rangle,$$

and the first term tends to zero in L^1 . By the general algebraic computation from the proof of Proposition 18, the second term becomes

$$\frac{1}{3} \langle c_n, c_n \wedge c_n \rangle \delta_{m\ell}.$$

We need to see that this tends to $\frac{1}{3} \langle c, (c \wedge c) \rangle \delta_{m\ell}$ in L^1 . Looking at the expressions for c_n and $c_n \wedge c_n$ from equations (48) and (49) we see that the only term that we have to worry about is $\langle a_n, d\varphi \wedge \xi_n \rangle$ because all other terms are products of factors in L^2 . For this remaining term we use the formula $a_n = \frac{1}{2} \varphi^{-1} d\varphi + \varphi \xi_n$ to see the magic improvement. Although $d\varphi$ is in L^2 and $d\varphi \wedge \xi_n$ is in $L^{3/2}$, the form $d\varphi \wedge d\varphi \wedge \xi_n$ converges in L^1 because φ has finite Faddeev energy. This completes the proof of equation (37).

Still assuming that ψ is smooth we now prove that we can pick a cartesian intertwining map Ψ with $\delta\langle\Psi^{-1}d\Psi, \psi\rangle = 0$. We look for the desired intertwining map in the form $\Psi = \Phi \mathbf{q}(\psi, \lambda)$. Recall that the map \mathbf{q} is defined by $\mathbf{q}(\psi, \lambda) = v^{-1}\lambda v$ where $\psi = v^{-1}\mathbf{i}v$. A direct computation shows that $\mathbf{q}(\psi, \lambda) \psi \mathbf{q}(\psi, \lambda)^{-1} = \psi$ so that Ψ is also an intertwining map for φ and ψ . We have

$$\Psi^{-1}d\Psi = \mathbf{q}(\psi, \lambda)^{-1}\Phi^{-1}d\Phi \mathbf{q}(\psi, \lambda) + \mathbf{q}(\psi, \lambda)^{-1}d\mathbf{q}(\psi, \lambda).$$

Using the invariance of the inner product under conjugation this gives

$$\langle\Psi^{-1}d\Psi, \psi\rangle = \langle\Phi^{-1}d\Phi, \psi\rangle + \langle\mathbf{q}(\psi, \lambda)^{-1}d\mathbf{q}(\psi, \lambda), \psi\rangle.$$

Now,

$$\mathbf{q}(\psi, \lambda)^{-1}d\mathbf{q}(\psi, \lambda) = v^{-1}dv - \mathbf{q}(\psi, \lambda)^{-1}v^{-1}dv \mathbf{q}(\psi, \lambda) + v^{-1}\lambda^{-1}d\lambda v.$$

This implies that

$$\langle\mathbf{q}(\psi, \lambda)^{-1}d\mathbf{q}(\psi, \lambda), \psi\rangle = \langle\lambda^{-1}d\lambda, \mathbf{i}\rangle. \quad (53)$$

Since $\langle\Phi^{-1}d\Phi, \psi\rangle$ is in $W^{1,2}$ by (46) and (45), we have the Hodge decomposition,

$$\langle\Phi^{-1}d\Phi, \psi\rangle = d\theta + \delta\alpha + \omega,$$

where θ is a $W^{2,2}$ function, α is a $W^{2,2}$ 2-form, and ω is a harmonic 1-form. We take $\lambda = e^{-i\theta}$ and compute

$$\langle\Psi^{-1}d\Psi, \psi\rangle = \langle\Phi^{-1}d\Phi, \psi\rangle + \langle\lambda^{-1}d\lambda, \mathbf{i}\rangle = \delta\alpha + \omega.$$

Thus $\delta\langle\Psi^{-1}d\Psi, \psi\rangle = 0$.

We need to check that the new map Ψ is still cartesian. To see this notice that we can apply the same argument that we used to prove that Φ was cartesian to Ψ the only change is that we will replace w by $\lambda^{-1}w$. This changes ξ to $\xi + d\theta$. This is still in $W^{1,2}$ and this is all we need to make the argument work.

To complete the proof of the theorem, we just need to compare different intertwining maps. Let Φ_1 and Φ_2 be two different maps intertwining φ and ψ . Then $\Phi_1^{-1}\varphi\Phi_1 = \Phi_2^{-1}\varphi\Phi_2$ implies that $\lambda := u_p\Phi_2\Phi_1^{-1}u_p^{-1}$ commutes with \mathbf{i} . Hence, $\lambda(x) \in S^1$ and $\lambda \in W^{1,2}(M, S^1)$. This expression for λ is independent of the local representative of φ exactly because any other local representative would have the form $\mu_p u_p$ and both μ_p and λ are complex. By assumption $\psi = \Phi_1^{-1}u_p^{-1}\mathbf{i}u_p\Phi_1$, so

$$\Phi_1\mathbf{q}(\psi, \lambda) = \Phi_1\Phi_1^{-1}u_p^{-1}\lambda u_p\Phi_1 = \Phi_2$$

(see (38)). Now $d\langle\lambda^{-1}d\lambda, \mathbf{i}\rangle = 0$. Assuming $\delta\langle\Phi_k^{-1}d\Phi_k, \psi\rangle = 0$ for $k = 1, 2$ we compute

$$0 = \delta\langle\Phi_2^{-1}d\Phi_2, \psi\rangle = \delta\langle\Phi_1^{-1}d\Phi_1, \psi\rangle + \delta\langle d\mathbf{q}(\psi, \lambda)\mathbf{q}(\psi, \lambda)^{-1}, \psi\rangle = \delta\langle\lambda^{-1}d\lambda, \mathbf{i}\rangle,$$

and conclude that $\langle\lambda^{-1}d\lambda, \mathbf{i}\rangle$ is harmonic. It follows that λ is smooth. \square

5 Integrality of the degree

The secondary homotopy invariant for a map from a 3-manifold to S^2 can be interpreted as the degree of a map. In this section we first discuss integrality results for the degrees of maps in general, then apply our discussion to finite Skyrme energy maps, and finally apply the techniques to construct a secondary homotopy invariant for finite Faddeev energy S^2 -valued maps. A smooth map $\Phi : P \rightarrow Q$ between closed, connected manifolds of the same dimension induces a map between the cohomology groups $\Phi^* : H^{\text{top}}(Q) \rightarrow H^{\text{top}}(P)$. As each of these groups is isomorphic to the integers this map is just multiplication by some integer. This integer is called the degree of the map. Using the deRham model one can write

$$\deg(\Phi) = \int_P \Phi^* \omega_Q,$$

where ω_Q is a normalized volume form on Q . This formula makes sense for sufficiently regular but possibly discontinuous Sobolev maps. For such maps it is interesting to ask whether the integral is still an integer.

Remark 29 *The map Φ from Example 2.4 is the identity on one half of a sphere and projects points along geodesics from a pole to the equator on the other half of the sphere. From this description or a direct computation one can see that $\int_{S^n} \Phi^* \omega_{S^n} = 1/2$ (in fact any real number can be achieved through a map with similar regularity). As we remarked in Example 2.4, $\Phi \in W^{1,p}$ for every $p < n$, and this is not enough regularity to conclude that the degree is an integer. This example is close to the borderline of the required regularity.*

Remark 30 *If $\Psi : P \rightarrow Q$ is a map between closed manifolds of the same dimension for which $\int_P \Psi^* \omega_Q$ is fractional, then Ψ cannot be approximated by smooth functions in any norm strong enough to imply convergence of the corresponding integrals, because this integral evaluated for any smooth function is integral. This implies a close relationship between integrality results and approximation results.*

Here is a useful general result to prove that the expression for the degree is an integer (see [32]).

Proposition 31 *Let $\Phi : P \rightarrow Q$ be a $W^{1,1}$ map between two closed manifolds of the same dimension. Let ω_Q be a smooth volume form on Q . Assume that $\Phi^* \omega_Q$ is integrable on P . In addition, assume, that Φ has the following property. For any $h \in L^\infty(Q)$ such that*

$$\int_Q h \omega_Q = 0, \tag{54}$$

the pulled back form $\Phi^(h \omega_Q)$ is integrable on P and*

$$\int_P \Phi^*(h \omega_Q) = 0. \tag{55}$$

Then

$$\int_P \Phi^*(\omega_Q) = \deg(\Phi) \int_Q \omega_Q, \quad (56)$$

where $\deg(\Phi)$ is an integer. If the map Φ is smooth, then $\deg(\Phi)$ coincides with the degree of Φ .

Proof. Under our assumptions, there exists an integrable function $N_\Phi(\cdot) : Q \rightarrow \mathbb{Z}$ such that for any scalar $h \in L^\infty(Q)$ we have the following area formula:

$$\int_P \Phi^*(h \omega_Q) = \int_Q h N_\Phi \omega_Q. \quad (57)$$

This formula is justified using the arguments of [22, 3.2.5, 3.2.20, 3.2.46], [47, Theorem 2], [36, Theorem 6.4]. We need to show that N_Φ is constant on Q . Assume that this function takes the values a and b on two sets of positive measure, say $N_\Phi^{-1}(a)$ and $N_\Phi^{-1}(b)$. Now choose

$$h(x) = \left(\int_{N_\Phi^{-1}(b)} \omega_Q \right) \mathbf{1}_{N_\Phi^{-1}(a)}(x) - \left(\int_{N_\Phi^{-1}(a)} \omega_Q \right) \mathbf{1}_{N_\Phi^{-1}(b)}(x).$$

This function satisfies equation (54) which, by assumption, implies (55). Thus

$$\left(\int_{N_\Phi^{-1}(a)} \omega_Q \right) \left(\int_{N_\Phi^{-1}(b)} \omega_Q \right) (a - b) = \int_Q h N_\Phi \omega_Q = \int_P \Phi^*(h \omega_Q) = 0.$$

Thus, $a = b$ and $N_\Phi(x)$ must be constant on Q . It is well known that N_Φ equals the degree of Φ for smooth Φ ; see e.g. [38]. \square

Remark 32 In applications, proving that (54) implies (55) amounts to proving that the differential d commutes with the pull-back Φ^* . Indeed, equation (54) shows that the closed form $h \omega_Q$ is exact, $h \omega_Q = d\alpha$. Hence, $\int_P \Phi^*(h \omega_Q) = \int_P \Phi^*(d\alpha)$, and if $\Phi^*d = d\Phi^*$, then $\int_P \Phi^*(d\alpha) = \int_P d(\Phi^*\alpha) = 0$. In general, pull-backs by Sobolev maps (of relatively low regularity) do not commute with d . So, the map, Φ , should have some special structure or additional integrability for this to happen. Cartesian maps have the required additional regularity. This is why we prove that there are cartesian intertwining maps.

We could have stated Proposition 31 using smooth functions of arbitrarily small support as well because of the following lemma.

Lemma 33 If $\Phi \in W^{1,1}(P, Q)$ is such that $\Phi^*\omega_Q \in L^1$ and Φ satisfies the implication

$$\int_Q f \omega_Q = 0 \quad \text{implies} \quad \int_P \Phi^*(f \omega_Q) = 0$$

for smooth functions f of arbitrarily small support, then it satisfies the implication for all functions in L^∞ .

Proof. It is not hard to show that any L^∞ function h on a closed manifold Q with $\int_Q h \omega_Q = 0$ can be written as a finite sum of L^∞ functions with arbitrarily small supports and each of which has zero average as well. Therefore, from the very beginning we will assume that the function $h \in L^\infty(Q)$ has conveniently small support on Q .

Given $h \in L^\infty(Q)$ with zero average, there exists a sequence h_k of C^∞ functions with zero average and such that they are uniformly bounded and converge to h almost everywhere. To see this, first mollify h to obtain C^∞ functions $\tilde{h}_\epsilon = \int \rho_\epsilon(t-t')h(t')dt'$ (we are in a small chart, i.e., basically in \mathbb{R}^n). Since $\tilde{h}_\epsilon \rightarrow h$ in L^1 , the averages $\langle \tilde{h}_\epsilon \rangle := \int_Q \tilde{h}_\epsilon \omega_Q$ go to zero. Pick a smooth function, ζ , with support in the same chart as h and with average 1. Now define $h_k = \tilde{h}_{\epsilon_k} - \langle \tilde{h}_{\epsilon_k} \rangle \zeta$, where $\epsilon_k \rightarrow 0$.

Now $\int_P \Phi^*(h_k \omega_Q) = 0$ by the assumed implication. Combining this with the assumption that $\Phi^* \omega_Q \in L^1$ and the dominated convergence theorem imply that

$$\int_P \Phi^*(h \omega_Q) = \int_P h(\Phi) \Phi^* \omega_Q = 0. \quad (58)$$

□

The following lemma establishes a special representation of average zero 3-forms on S^3 that explains our interest in commuting pull-back and exterior differentiation applied to forms $\langle f, y^{-1}dy \wedge y^{-1}dy \rangle$.

Lemma 34 *Given h a smooth real-valued function on S^3 with average 0:*

$$\int_{S^3} h \omega_{S^3} = 0,$$

there exists a \mathbb{R}^3 -valued smooth function f on S^3 such that

$$h \omega_{S^3} = d\langle f, y^{-1}dy \wedge y^{-1}dy \rangle.$$

Proof. For any scalar function g on S^3 we have

$$dg = X_1(g) \theta^1 + X_2(g) \theta^2 + X_3(g) \theta^3,$$

where $\theta = y^{-1}dy$, X_k and θ^k are the Lie algebra-valued forms, vector fields and forms introduced in Proposition 18. Thus, for any function $f(y) = \mathbf{i}f^1(y) + \mathbf{j}f^2(y) + \mathbf{k}f^3(y)$,

$$d\langle f, \theta \wedge \theta \rangle = \langle df, \theta \wedge \theta \rangle = \sum X_i(f^i) \theta^1 \wedge \theta^2 \wedge \theta^3.$$

Choose

$$f^i = X_i(u),$$

where u is a solution of the Poisson equation

$$L(u) = -\frac{1}{2\pi^2} h.$$

The corresponding f is the desired function because the Laplace-Beltrami operator is given by $L := -\sum_{i=1}^3 X_i^2$ and the volume form is given by $\omega_{S^3} = \frac{1}{2\pi^2} \theta^1 \wedge \theta^2 \wedge \theta^3$. □

5.1 Degree for finite Skyrme energy maps

Our first application of Proposition 31 shows that the degree of a finite Skyrme energy map from a closed 3-manifold into S^3 is integral. Integrality was proved in [18] for finite Skyrme energy maps on \mathbb{R}^3 by the method outlined in Proposition 31 and Remark 32. It was also proved for $W^{1,3}$ maps on \mathbb{R}^3 by Rivière, [41].

Proposition 35 *If the map $u \in W^{1,2}(M, S^3)$ has a finite Skyrme energy or $u \in W^{1,3}(M, S^3)$ then $\int_M u^* \omega_{S^3}$ is an integer.*

Remark 36 *We define the degree of a finite Skyrme energy map to be the integer given by this proposition. This extends the usual notion of degree from smooth maps.*

Proof. Combining Lemma 34, Lemma 33 and Proposition 31 we see that it is sufficient to prove that

$$u^* (d\langle f, y^{-1} dy \wedge y^{-1} dy \rangle) = d(u^* (\langle f, y^{-1} dy \wedge y^{-1} dy \rangle)) .$$

For any finite Skyrme energy map $u : M \rightarrow S^3$ both $u^{-1} du$ and $u^{-1} du \wedge u^{-1} du$ are in L^2 and a straightforward application of the approximation (by mollification, T_ϵ) argument gives:

$$\begin{aligned} d(u^* (\langle f, y^{-1} dy \wedge y^{-1} dy \rangle)) &= \lim_{\epsilon \rightarrow 0} d(\langle T_\epsilon(u^* f), T_\epsilon(u^{-1} du \wedge u^{-1} du) \rangle) \\ &= \lim_{\epsilon \rightarrow 0} (\langle dT_\epsilon(u^* f), T_\epsilon(u^{-1} du \wedge u^{-1} du) \rangle + \langle T_\epsilon(u^* f), dT_\epsilon(u^{-1} du \wedge u^{-1} du) \rangle) \\ &= u^* (d\langle f, y^{-1} dy \wedge y^{-1} dy \rangle) . \end{aligned}$$

For a $W^{1,3}$ map, the same approximation argument still works. □

The result of the previous proposition generalizes to Chern-Simons invariants. In this case the set of all possible values is unknown even in the smooth case (It is conjectured that all such Chern-Simons invariants are rational.)

Corollary 37 *If A is a finite Skyrme energy, $\int_M |A|^2 + |A \wedge A|^2 < \infty$, distributionally flat $SU(2)$ connection, then there is a smooth connection with the same holonomy and Chern-Simons invariant.*

Proof. It is well known that any representation is the holonomy of a smooth connection. By [5, Lemma 6] two flat connections with the same holonomy are gauge equivalent, so we may assume that our finite energy flat connection is gauge equivalent to a smooth reference connection. The Chern-Simons invariants of two gauge equivalent connections are related by the degree of the gauge transformation. See the expression above [5, Lemma 7]. By Proposition 35 this degree is an integer for the finite energy gauge transformation. Finally we can change the Chern-Simons of the smooth reference connection by this integer by a suitable smooth gauge transformation. □

Remark 38 *The same results also hold for maps $u : M \rightarrow S^3$ and flat connections in $W^{1,3}$. In the next subsection we address maps to S^2 in the finite Faddeev energy case. All of our results are also valid for maps in $W^{1,3}$.*

5.2 The secondary invariant for S^2 -valued maps

In this subsection we justify the definition of our numerical secondary homotopy invariant (16) for finite Faddeev energy maps. Notice that we know that this integral converges by Theorem 28. We will have to prove that this integral is an integer for sufficiently regular Sobolev maps. This integrality result follows the general outline described at the start of the section.

Proposition 39 *If $\varphi : M \rightarrow S^2$ is a finite Faddeev energy map, $\phi : M \rightarrow S^2$ is a smooth map, $\varphi^* \mu_{S^2} = \phi^* \mu_{S^2}$ and $\Phi : M \rightarrow S^3$ is a cartesian map satisfying $\varphi = \Phi \phi \Phi^{-1}$ and $\delta \langle \Phi^{-1} d\Phi, \phi \rangle = 0$ as given by Theorem 28, then*

$$\Upsilon(\varphi, \phi) := \deg(\Phi) := -\frac{1}{12\pi^2} \int_M \operatorname{Re}((\Phi^{-1} d\Phi)^3)$$

is an integer. This is valid for M any closed 3-manifold or \mathbb{R}^3 .

Proof. Combining Lemma 34, Lemma 33, Proposition 31 and Theorem 28 gives the result for closed manifolds M . Using Theorem 22 gives the result for \mathbb{R}^3 . \square

We will see that changing the intertwining map only changes $\Upsilon(\varphi, \phi)$ by a multiple of $2m_\varphi$ where m_φ is the divisibility of $\varphi^* \mu_{S^2}$.

Definition 40 *The divisibility of a class $\beta \in H^2(M; \mathbb{Z})$ is the unique non-negative integer m such that*

$$(\beta \cup H^1(M; \mathbb{Z})) \cap [M] = m\mathbb{Z}.$$

This motivates the following definition of a secondary homotopy invariant for finite Faddeev energy maps.

Definition 41 *Given two finite Faddeev energy maps φ and ψ with $\varphi^* \mu_{S^2} = \psi^* \mu_{S^2}$ let m_φ be the divisibility of the class $\varphi^* \mu_{S^2}$. Let ϕ be a smooth map with the same primary invariant as φ and define the secondary invariant by*

$$\Upsilon(\varphi, \psi) = \Upsilon(\varphi, \phi) - \Upsilon(\psi, \phi) \pmod{2m_\varphi}.$$

For $M = \mathbb{R}^3$ there is no primary invariant so we take $\psi = \phi = \mathbf{i}$ and write $\Upsilon(\varphi)$.

Remark 42 *It is obvious that this definition agrees with the numerical invariant from equation (16) on smooth maps – just take $\phi = \psi$ and pick the constant map 1 as the intertwining map between ϕ and ψ . The difference in sign from equation (16) is explained by noticing that the result computed with Φ is minus the result computed with Φ^{-1} . Comparing the direction of the intertwining, we see that we have the correct sign. It is also clear from the definition that $\Upsilon(\varphi, \psi) = -\Upsilon(\psi, \varphi)$ and*

$$\Upsilon(\varphi, \psi) = \Upsilon(\varphi, \chi) + \Upsilon(\chi, \psi).$$

The remainder of this section will demonstrate that this secondary invariant is independent of the choice of smooth map ϕ and gauge-fixed cartesian intertwining maps (Theorem 28). The main ingredient is the following formula for the degree of a product of two S^3 -valued functions:

$$\deg(\Phi_1 \Phi_2) = \deg(\Phi_1) + \deg(\Phi_2).$$

This formula follows by a simple application of the product rule to $\Phi_3 = \Phi_1 \Phi_2$ giving

$$\operatorname{Re}((\Phi_3^{-1}d\Phi_3)^{\wedge 3}) = \operatorname{Re}((\Phi_1^{-1}d\Phi_1)^{\wedge 3}) + \operatorname{Re}((\Phi_2^{-1}d\Phi_2)^{\wedge 3}) - 3d \operatorname{Re}(\Phi_1^{-1}d\Phi_1 \wedge d\Phi_2 \Phi_2^{-1}).$$

In concrete applications we just have to make sure that we have enough regularity to apply the product rule to $\Phi_1 \Phi_2$ and $\Phi_1^{-1}d\Phi_1 \wedge d\Phi_2 \Phi_2^{-1}$.

Lemma 43 *The secondary invariant $\Upsilon(\varphi, \psi)$ is independent of the choice of intermediate smooth map ϕ .*

Proof. Let χ be a smooth map with the same primary invariant as ϕ . By Theorem 10 there is a smooth intertwining map Ψ so that $\phi = \Psi\chi\Psi^{-1}$. Let Φ be the intertwining map ($\varphi = \Phi\phi\Phi^{-1}$) used to compute $\Upsilon(\varphi, \phi)$. By Theorem 28 Φ is in $W^{1,2}$ and $(\Phi^{-1}d\Phi)^{\wedge 2}$ is in $L^{3/2}$. Clearly $\Phi\Psi$ is an intertwining map for χ and φ . That the product rule is valid on $\Phi\Psi$ follows from the approximation argument using the fact that Φ is in $W^{1,2}$. That the product rule is valid on $\Phi^{-1}d\Phi \wedge d\Psi\Psi^{-1}$ follows from the approximation argument using the fact that $(\Phi^{-1}d\Phi)^{\wedge 2}$ is in $L^{3/2}$ and $\Phi^{-1}d\Phi$ is in L^1 . This implies that

$$\Upsilon(\varphi, \chi) = \Upsilon(\varphi, \phi) + \deg(\Psi).$$

The same formula applies to $\Upsilon(\psi, \phi)$ so the $\deg(\Psi)$ terms cancel. \square

Remark 44 *The intermediate smooth map was introduced exactly because we do not have an argument validating the product rule applied to $\Phi^{-1}d\Phi \wedge d\Psi\Psi^{-1}$ when the two S^2 -valued maps are only assumed to have finite energy. We expect that there is a direct argument validating the product rule but it will require some subtle cancelation.*

In order to understand the dependence of $\Upsilon(\varphi, \psi)$ on the choice of intertwining maps we recall that any intertwining map may be obtained from a fixed one as $\Phi \mathfrak{q}(\psi, \lambda)$. Thus we have to study the degree of the map $\mathfrak{q}(\psi, \lambda)$. We begin with an elementary calculation of the degree of the map $\mathfrak{q} : S^2 \times S^1 \rightarrow S^3$.

Lemma 45 *The map $\mathfrak{q} : S^2 \times S^1 \rightarrow S^3$ has degree two.*

Proof. Recall that $\mathfrak{q}(x, z) = u^{-1}zu$ where $x = u^{-1}\mathbf{i}u$. From equation (15) we have

$$\mathfrak{q}^*\omega_{S^3} = -\frac{1}{8\pi}x[a, x]^2\frac{1}{2\pi}\langle a, x \rangle,$$

where

$$a = \mathfrak{q}^{-1}d\mathfrak{q} = u^{-1}z^{-1}dz u + u^{-1}du - \mathfrak{q}^{-1}u^{-1}du\mathfrak{q}.$$

The term $u^{-1}z^{-1}dz u$ is parallel to x and the remainder of a is perpendicular to x so

$$\frac{1}{2\pi}\langle a, x \rangle = \frac{1}{2\pi\mathbf{i}}z^{-1}dz = \omega_{S^1}.$$

Now

$$x[a, x]^2 = x[u^{-1}du, x]^2 + x[\mathbf{q}^{-1}u^{-1}du\mathbf{q}, x]^2 - x[u^{-1}du, x][\mathbf{q}^{-1}u^{-1}du\mathbf{q}, x] - x[\mathbf{q}^{-1}u^{-1}du\mathbf{q}, x][u^{-1}du, x].$$

Since \mathbf{q} commutes with x the second term is equal to the first term. Since x and $u^{-1}du$ anticommute

$$x[u^{-1}du, x][\mathbf{q}^{-1}u^{-1}du\mathbf{q}, x] = [u^{-1}du, x]x[\mathbf{q}^{-1}u^{-1}du\mathbf{q}, x] = x[\mathbf{q}^{-1}u^{-1}du\mathbf{q}, x][u^{-1}du, x].$$

It follows that $x[a, x]^2 = 2x[u^{-1}du, x]^2 = -16\pi\omega_{S^2}$ by equations (10) and (4). This completes the proof. \square

By Theorem 28 any two gauge fixed cartesian intertwining maps are related by

$$\Phi_1 = \Phi_2 \mathbf{q}(\phi, \lambda),$$

for some smooth map $\lambda : M \rightarrow S^1$. Notice that the computation in the proof of Lemma 45 is completely algebraic. This implies that

$$(\mathbf{q}(\phi, \lambda))^* \omega_{S^3} = 2\phi^* \omega_{S^2} \wedge \lambda^* \omega_{S^1}. \quad (59)$$

Now consider the map taking $C^\infty(M, S^1)$ to \mathbb{R} given by

$$\lambda \mapsto \int_M \phi^* \omega_{S^2} \wedge \lambda^* \omega_{S^1}.$$

Since $(\lambda\mu)^{-1}d(\lambda\mu) = \lambda^{-1}d\lambda + \mu^{-1}d\mu$ this map is a group homomorphism. This is just the degree of the smooth map $(\phi, \lambda) : M \rightarrow S^2 \times S^1$ which is also $(\phi^* \mu_{S^2} \cup \lambda^* \mu_{S^1}) \cap [M]$. The image of this map is a subgroup of \mathbb{Z} so it takes the form $m_\phi \mathbb{Z}$ for some non-negative m_ϕ . Since every first cohomology class can be represented by $\lambda^* \mu_{S^1}$ we see that the number m_ϕ is exactly the divisibility of $\phi^* \mu_{S^2} = \varphi^* \mu_{S^2}$. To finish we just have to verify that the formula for the degree of the product of Φ and $\mathbf{q}(\phi, \lambda)$ is valid.

Lemma 46 *The secondary invariant $\Upsilon(\varphi, \psi)$ is independent of the choice of intertwining maps.*

Proof. Since Φ is in $W^{1,2}$ and $\mathbf{q}(\phi, \lambda)$ is smooth and $(\Phi^{-1}d\Phi)^{\wedge 2}$ is in $L^{3/2}$ the standard approximation argument shows that the product rules for $d(\Psi \mathbf{q}(\phi, \lambda))$ and $d(\Phi^{-1}d\Phi \wedge d\mathbf{q}(\phi, \lambda) \mathbf{q}(\phi, \lambda)^{-1})$ are valid so the degree of $\Phi \mathbf{q}(\phi, \lambda)$ is just the sum of the degrees of Φ and $\mathbf{q}(\phi, \lambda)$ as required. \square

Notice that for \mathbb{R}^3 the secondary invariant $\Upsilon(\varphi)$ is well-defined and is an integer. The point is that the intertwining maps are well-defined up to multiplication by a constant and any constant map has degree zero. As we saw in Theorem 11 when $\varphi^* \mu_{S^2} = 0$ there is a direct expression for $\Upsilon(\varphi)$ that does not require an intertwining map. The arguments that we have given combine to show that this is an integer for finite energy maps as well.

Corollary 47 *If $\varphi : M \rightarrow S^2$ has finite Faddeev energy and $M = \mathbb{R}^3$ or $\varphi^*\mu_{S^2} = 0$ then there is a 1-form on M , call it θ , such that $\varphi^*\omega_{S^2} = d\theta$ and*

$$\text{Hopf}(\varphi) = \int_M \theta \wedge d\theta,$$

is a well-defined integer.

Proof. We just take $\theta = \frac{1}{2\pi} \langle \Phi^{-1}d\Phi, \varphi \rangle$ where Φ is the gauge fixed intertwining map. The remainder of the argument is just the argument from Theorem 11 with $N = 1$ and $\alpha = 0$.

6 Conclusion and applications

We conclude by addressing questions that were posed in [5], [6] and [33]. The first questions raised in [6] were about the regularity of minimizers of the Faddeev functional. This is still an interesting open question. The next question was how to extend obstruction theory for finite energy maps; what cohomology theory should one use? how would one define the primary invariant of a finite energy map? We have given a very satisfactory answer to these questions in Definition 25 and Theorem 28. We have defined the pull-back on cohomology for finite energy maps in a way that generalizes the pull-back for smooth maps. In addition we showed that the pull-back of the fundamental class on S^2 is the obstruction to the existence of a lift from S^2 to S^3 even for finite energy maps. Theorem 28 generalizes [6, Lemma 1] from the smooth case as it should. We gave an example of a discontinuous finite energy map that is not in VMO (Example 2.4) so our results are true, but not for trivial reasons.

Unlike the primary homotopy invariant for maps $\varphi : M \rightarrow S^2$, it was already clear from [6] how to generalize the secondary invariant for pairs of such maps once the primary invariant was defined. Likewise it was clear from [5] how to define the primary invariant for maps $u : M \rightarrow S^3$ or the Chern-Simons invariants of $SU(2)$ -connections. Each of these could be expressed as an integral and the integral could, in principle converge for possibly discontinuous finite energy maps. What was not clear is if the values taken by these integrals for finite-energy maps would coincide with the values taken by smooth maps. This left open the possibility of phantom sectors of finite energy maps containing no smooth maps. Proposition 35 and Corollary 37 demonstrate that this does not occur for S^3 . The generalization to other Lie groups is still an interesting open question.

Whereas the integrality of the Hopf invariant is clear for smooth maps, it was an open question for finite energy maps. The Hopf invariant is just a special case of the secondary invariant appearing in Pontrjagin's theorem. Proposition 39 establishes that the integral for the secondary invariant does indeed give an integer. Lemma 43, Lemma 46 and Lemma 45 show that the integral for the secondary invariant descends to give a well-defined invariant in the appropriate cyclic group for finite energy maps.

All of the results that we have discussed thus far would follow easily from an approximation theorem proving that any finite energy map could be approximated by smooth maps in a reasonable sense. Unfortunately this is not known. Smooth approximation remains an

interesting question. All of the examples of functions that are not reasonably approximated by smooth functions that we know arise because they do not respect homotopy properties satisfied by smooth maps (for example have fractional degree). We have seen that finite energy maps satisfy the properties that one would expect based on the homotopy theory of smooth maps, so we expect that there is a reasonable way to approximate finite Skyrme and Faddeev energy maps.

In [6], we proved the existence of minimizers of the Faddeev functional in a slightly bizarre looking function space. The results of this paper show that the function space was just the space of finite energy Sobolev maps in a fixed homotopy class.

Proposition 48 *Let M be a closed orientable 3-manifold. For any finite energy $\varphi : M \rightarrow S^2$, the functional*

$$E(\psi) = \int_M |d\psi|^2 + |d\psi \wedge d\psi|^2 d\text{vol}_M.$$

has a minimizer in the class

$$\{\psi \in W^{1,2}(M, S^2) | E(\psi) < \infty, \varphi^* \mu_{S^2} = \psi^* \mu_{S^2}, \Upsilon(\varphi, \psi) = 0 \pmod{2m_\varphi}\}.$$

Furthermore this class contains a smooth map.

Proof. Given a finite energy map φ we know from Theorem 28 that there is a smooth map ϕ with the same primary invariant. By Proposition 39 we know that $\Upsilon(\varphi, \phi)$ is an integer. We also know that every integer is the degree of some smooth map from M to S^3 (just compose the map obtained by collapsing the two-skeleton with a degree n self map of S^3). Let Φ be a map of degree $\Upsilon(\varphi, \phi)$ so that there is a smooth map $\chi := \Phi\phi\Phi^{-1}$ in the class with

$$\Upsilon(\varphi, \chi) = \Upsilon(\varphi, \phi) - \deg(\Phi) = 0.$$

By the main existence result from [6] we know that there is a minimizer of E in the class of functions denoted by \mathfrak{A}_χ in [6]. This class of functions consists of functions $\psi = \Psi^{-1}\chi\Psi$ such that

1. $E(\psi) < \infty$
2. $\Upsilon(\psi, \chi) = 0 \pmod{2m_\varphi}$
3. $\delta\langle\Psi^{-1}d\Psi, \chi\rangle = 0$
4. $\mathcal{H}\langle\Psi^{-1}d\Psi, \chi\rangle = \sum_k h_k \eta_k,$

where \mathcal{H} is the harmonic projection of the form, $h_k \in [0, 1]$ and η_k are an integral basis for the image of $H^1(M; \mathbb{Z})$ under \mathcal{H} in the space of harmonic 1-forms. We will see that these two classes of functions coincide. Indeed, Theorem 28 tells us that ϕ and φ are related by an intertwining map in $W^{1,2}$ since they have the same primary invariant, hence φ and ψ are related by an intertwining map obtained as the product of the three intertwining maps. This

implies that any function in \mathfrak{A}_χ has the same primary invariant as φ by Theorem 28. The finite energy condition is given by item 1 and the condition on the secondary invariant is given by item 2. Working in the other direction, we see that conditions 1, 2 and 3 hold from the assumptions on the class described in this theorem together with the construction of χ provided that a gauge-fixed intertwining map is chosen. To obtain the fourth condition just notice that changing the intertwining map by right multiplication by $\mathfrak{q}(\chi, \lambda)$ with harmonic λ preserves the first three conditions and changes the value of $\mathcal{H}\langle \Psi^{-1}d\Psi, \chi \rangle$ by an arbitrary integral harmonic form. \square

Remark 49 *The assumption that M is orientable is really not necessary. We could look for \mathbb{Z}_2 -equivariant minimizers on the orientation cover. In this case there would be no secondary invariant.*

Similar questions about classes of functions are raised by S. Koshkin in [33]. His paper addresses minimization of the Faddeev functional for maps into homogeneous and symmetric spaces. In [33] the class of admissible maps \mathcal{E} is introduced to study the minimization problem in a fixed 2-homotopy sector. The class \mathcal{E} is defined as the maps $\psi = u^{-1}\varphi u$ where φ is smooth and $a^\perp \in L^2$, $a^\perp \wedge a^\perp \in L^2$ and $a^\parallel \in W^{1,2}$ where $a = u^{-1}du$ and in our case $a^\parallel = \langle a, \varphi \rangle \varphi$, $a^\perp = \frac{1}{2}\varphi[a, \varphi]$. Koshkin conjectured [33, Conjecture 1] that the union of the \mathcal{E} taken over a family of smooth functions φ representing every homotopy type is exactly the class of finite energy maps. We can now see that this conjecture is true for S^3/S^1 . Indeed let ψ be a finite energy map and let φ be a smooth map with the same primary invariant as given by Proposition 27. Setting Φ to be the gauge-fixed intertwining map given by Theorem 28 we have $\varphi = \Phi^{-1}\psi\Phi$ so

$$a^\perp = \frac{1}{2}\varphi[a, \varphi] = \frac{1}{2}\varphi\Phi^{-1}d\psi\Phi - \frac{1}{2}\varphi d\varphi,$$

and

$$a^\parallel = \langle a, \varphi \rangle \varphi = \langle u^{-1}du, \varphi \rangle \varphi - \langle v^{-1}dv, \psi \rangle \varphi,$$

where u and v are local lifts of φ and ψ respectively. The first term in a^\parallel is smooth and with the proper gauge-fixing the second term will be in $W^{1,2}$. The first term in a^\perp is in L^2 since ψ has finite energy and Φ is bounded and the second term is smooth. The only term that one needs to worry about in $a^\perp \wedge a^\perp$ is the square of the first term in a^\perp . This is just

$$\frac{1}{4}\varphi\Phi^{-1}d\psi\Phi\varphi\Phi^{-1}d\psi\Phi = -\frac{1}{4}\varphi\Phi^{-1}\psi d\psi \wedge d\psi\Phi,$$

which is also in L^2 since ψ has finite energy. Working in the other direction uses the same formulas to see that ψ has finite energy given that $a^\perp \in L^2$, $a^\perp \wedge a^\perp \in L^2$ and $a^\parallel \in W^{1,2}$. The argument is contained in [6] and [33].

Versions of our local representation result (Proposition 18), primary invariant (Definition 25) and global intertwining result (Theorem 28) should hold for homogeneous spaces of the form G/T^k with only minor modifications. This class includes all flag manifolds. It follows

that [33, Conjecture 1] also holds for these spaces. The definition of the primary invariant becomes more interesting for homogeneous spaces G/H when H is nonabelian. One can not take cohomology with values in H as we did in the proof of Theorem 28, but this is probably not needed. The point is that the primary invariant will arise as a cocycle analogous to our n but this should be interpreted as the degree of various overlap maps into H .

Appendix 1: The $W^{1,1}$ image of a connected set

The question whether the image of a connected set under a Sobolev map is connected or not has been discussed in the literature. For $W^{1,1}$ maps, there is a result of Giaquinta, Modica and Souček, [23], that the image, $\bar{u}(\mathcal{A}_C(u))$,¹ of the set of approximate continuity, $\mathcal{A}_C(u)$, of a map $u \in W^{1,1}(\Omega, \mathbb{R}^N)$ on an open connected set $\Omega \subset \mathbb{R}^n$, is connected; see [23] for details. This would suffice for our purposes, but we give a simple independent argument proving a convenient substitute connected image result for Sobolev maps.

Lemma 50 *Let X be a smooth, compact, connected manifold of dimension n , and let Y be a compact subset of \mathbb{R}^N . For any map $f \in W^{s,p}(X; Y)$, where $s > 0$, $p \geq 1$, and $s - \frac{n}{p} \geq 1 - n$, there exists a connected component, Y_1 , of Y such that $f(x) \in Y_1$ for almost all $x \in X$.*

Proof. In view of the embedding $W^{s,p} \subset W^{1,1}$, it is sufficient to consider only the case $f \in W^{1,1}$. Using the description of Sobolev spaces on closed manifolds, choose a finite open cover $\{U_i\}$ of X together with smooth coordinate maps $\chi_i : U_i \rightarrow 2I^n$ with the property that the open sets $\chi_i^{-1}(I^n)$ still cover X . Let ϑ be a smooth cut-off function on X with support in $\chi_i^{-1}(2I^n)$, which is positive on the set $\chi_i^{-1}(\frac{3}{2}I^n)$ and, in addition, $\vartheta(p) = 1$ for $p \in \chi_i^{-1}(I^n)$. Given $f \in W^{s,p}(X; Y)$, the function $(\vartheta \cdot f) \circ \chi_i^{-1}$ belongs to $W^{s,p}(2I^n; \mathbb{R}^N)$ and has the property that when restricted to the cube I^n its values lie in Y . If we prove, that, in fact, its values are in one connected component of Y , then applying this argument to each chart and using the connectedness of X we prove the lemma. Thus, we localize our consideration to a cube and prove the following result:

If $f \in W^{1,1}(I^n(1); \mathbb{R}^N)$ is such that $f(x) \in Y$ for a.e. $x \in I^n(1)$, then, except for at most a set of measure 0, the image of the cube $I^n(1)$ under the map f lies entirely in one component of Y .

The proof is by induction on n , the dimension of the cube. If $n = 1$, the statement of the lemma is true because any function in $W^{1,1}(I^1(1); \mathbb{R})$ is absolutely continuous after possibly a change on a set of measure zero ([52]). Assuming the lemma is true for all maps in $W^{1,1}(I^{n-1}(1); \mathbb{R}^N)$, let us prove it for an $f \in W^{1,1}(I^n(1); \mathbb{R}^N)$. Split the coordinates x^1, \dots, x^n into $x' = (x^1, \dots, x^{n-1})$ and x^n . It is not hard to see that (after a possible change on a set of measure zero in I^n) $f(\cdot, t) \in W^{1,1}(I^{n-1}; \mathbb{R}^N)$, for a.e. $t \in I^1$. Moreover, there is

¹where $\bar{u}(x) = \text{aplim}_{y \rightarrow x} u(y)$ for $x \in \mathcal{A}_C(u)$

a set $\Theta \in I^1$ of measure 1 such that, for every $t \in \Theta$, not only $f(\cdot, t) \in W^{1,1}(I^{n-1}(1); \mathbb{R}^N)$, but also $f(x', t) \in Y$, for almost all $x' \in I^{n-1}(1)$. On the other hand, it is well known (see [52], 2.1.4) that a function in $W^{1,1}$ is absolutely continuous on almost every line segment parallel to the coordinate axes. Thus, there is a set $\Xi \in I^{n-1}(1)$ of measure 1 such that $f(x', \cdot)$ is absolutely continuous for all $x' \in \Xi$. At this point we stop adjusting f .

For every $t \in \Theta$, by the induction assumption, there is a set $\Sigma_t \in I^{n-1}(1)$ of full measure and a connected component Y_t of Y such that $f(x', t) \in Y_t$, for all $x' \in \Sigma_t$. Assume that for some t_1 and t_2 the corresponding components Y_{t_1} and Y_{t_2} are different. Take any x' from the intersection $\Sigma_{t_1} \cap \Sigma_{t_2} \cap \Xi$ (which has measure 1). Then $f(x', x^n)$, for all $x^n \in I^1$, should lie in the same component of Y . A contradiction. This shows that all Y_t are the same and proves the lemma.

Remark 51 *Following Brezis and Nirenberg [16], define the essential range, $\text{essRange}(f)$, of a measurable map f from a compact set $X \subset \mathbb{R}^n$ into a compact set $Y \subset \mathbb{R}^N$ as the smallest closed set \mathfrak{C} in Y such that $f(x) \in \mathfrak{C}$ for almost every $x \in X$. The essential range is well defined, as shown in [16]. Proposition 24 shows that the essential range of a $W^{1,1}$ map from a connected manifold into a compact subset of \mathbb{R}^N lies entirely in one connected component of the target set.*

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